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SINGULAR LIMIT OF SOLUTIONS OF $u_t = \Delta u^m - A \cdot \nabla(u^q/q)$ AS $q \to \infty$

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ABSTRACT. We will show that the solutions of $u_t = \Delta u^m - A \cdot \nabla(u^q/q)$ in $R^n \times (0, T)$, $T > 0$, $m > 1$, $u(x, 0) = f(x) \in L^1(R^n) \cap L^\infty(R^n)$ converge weakly in $(L^\infty(G))^*$ for any compact subset G of $R^n \times (0, T)$ as $q \to \infty$ to the solution of the porous medium equation $v_t = \Delta v^m$ in $R^n \times (0, T)$ with $v(x, 0) = g(x)$ where $g \in L^1(R^n)$, $0 \le g \le 1$, satisfies $g(x) + (\tilde{g}(x))_{x_1} = f(x)$ in $\mathscr{D}'(R^n)$ for some function $\tilde{g}(x) \in L^1(R^n)$, $\tilde{g}(x) \ge 0$ such that $g(x) = f(x)$, $\tilde{g}(x) = 0$ whenever $g(x) < 1$ a.e. $x \in R^n$. The convergence is uniform on compact subsets of $R^n \times (0, T)$ if $f \in C_0(R^n)$.

In this paper we will study the asymptotic behaviour of nonnegative solutions $u = u^{(q)}$ of the equation

$$(0.1) \qquad \begin{cases} u_t = \Delta u^m - A \cdot \nabla(u^q/q), & (x, t) \in R^n \times (0, T), \\ u(x, 0) = f(x) \ge 0, & f \in L^1(R^n) \cap L^\infty(R^n), \end{cases}$$

where $0 \neq A = (a_1, a_2, \dots, a_n) \in R^n$ is a constant vector, $T > 0$, $m > 1$, as $q \to \infty$. Recently there is a lot of research on the above equation ([A],[DiK],[G1],[G2]) The equation arises in many physical applications such as the flow of water through a homogeneous isotropic rigid porous medium [G1]. When $A = 0$, the above equation reduces to the well-known porous medium equation ([Ar],[P]). In the paper [CF], Caffarelli and A. Friedman studied the asymptotic behaviour of solutions of (0.1) when $A = 0$ and showed that the solutions of (0.1) converge as $m \to \infty$ if f satisfies (0.1) and the following conditions:

$$\begin{aligned} &f \in C^1 \text{ in supp } f, \\ &f(0) > 1, \quad f_r < 0 \text{ in } R^n \setminus \{0\} \cap \text{supp } f, \\ &f_{r_{x_0}} \le 0 \text{ in } R^n \setminus B_1(0) \cap \text{supp } f \qquad \forall x_0 \in B_{\varepsilon_0}(0) \end{aligned}$$

for some $\varepsilon_0 > 0$ where $r_{x_0} = |x - x_0|$, $B_r(0) = \{x : |x| < r\}$ and $f_{r_{x_0}}$ is the radial derivative of f with center at x_0.

This result has been extended in various directions by P. Bénilan, L. Boccardo and M. Herrero [BBH], P. E. Sacks [S2] in the case $A = 0$, X. Xu [X] in the case of hyperbolic equations and K. M. Hui [H1], [H2] in the case of a porous

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medium equation with absorption and in the case of the generalized p-Laplacian equation.

For simplicity we will assume that $T = 1$ and $A = (1, 0, \ldots, 0)$ throughout the rest of the paper. We will show that as $q \to \infty$, the convection term in (0.1) disappears. More precisely, we will show that for fixed $m > 1$ the solutions $u = u^{(q)}$ of (0.1) converge weakly in $(L^\infty(G))^*$ for any compact subset G of $R^n \times (0, 1)$ as $q \to \infty$. Moreover the limit $u^{(\infty)} = \lim_{q\to\infty} u^{(q)}$ satisfies the porous medium equation

$$\text{(0.2)} \qquad \begin{cases} v_t = \Delta v^m, & (x, t) \in R^n \times (0, 1), \\ v(\cdot, t) \searrow g & \text{as } t \to 0 \text{ in } \mathscr{D}'(R^n), \end{cases}$$

where $g \in L^1(R^n)$, $0 \le g \le 1$, satisfies

$$\text{(0.3)} \qquad g(x) + (\tilde{g}(x))_{x_1} = f(x) \text{ in } \mathscr{D}'(R^n)$$

for some function $\tilde{g}(x) \ge 0$, $\tilde{g}(x) \in L^1(R^n)$ and $g(x) = f(x)$, $\tilde{g}(x) = 0$ whenever $g(x) < 1$ a.e. $x \in R^n$. This extends the recent results obtained by M. Escobedo and E. Zuazua [EZ], who showed that the convection term was negligible compared with the other terms appearing in (0.1) for the case $m = 1$ and $q > 1 + 1/n$ as $t \to \infty$. Although we were not able to prove it, we suspect that the same result should remain valid when $A = A(x) \in L^\infty(R^n)$.

We will first start with some definitions. For any open set $\Omega_0 \subset R^n$, $h \in C(R)$, we say that u is a solution (respectively subsolution, supersolution) of

$$\text{(0.4)} \qquad u_t = \Delta u^m - (h(u))_{x_1}$$

in $\overline{\Omega}_0 \times (0, 1)$ if u is continuous and nonnegative in $\overline{\Omega}_0 \times (0, 1)$, $u \in L^\infty([0, 1); L^1(\Omega_0)) \cap L^\infty(\Omega_0 \times (0, 1))$ and satisfies

$$\text{(0.5)} \quad \int_{\tau_1}^{\tau_2} \int_\Omega \left[u^m \Delta\eta + u\frac{\partial \eta}{\partial t} + h(u)\eta_{x_1} \right] dx\,dt = \int_{\tau_1}^{\tau_2} \int_{\partial\Omega} u^m \frac{\partial \eta}{\partial N} d\sigma\, ds + \int_\Omega u\eta\, dx \bigg|_{\tau_1}^{\tau_2}$$

(respectively $\ge$, $\le$) for all bounded open sets $\Omega \subset \Omega_0$ with $\partial\Omega \in C^2$, $0 < \tau_1 \le \tau_2 < 1$, $\eta \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\eta \equiv 0$ on $\partial\Omega \times [\tau_1, \tau_2]$ where $\partial/\partial N$ is the exterior normal derivative on $\partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$.

If u is a solution of (0.4) in $\overline{\Omega}_0 \times (0, 1)$, we say that u has initial trace or initial value $d\mu$ if

$$\lim_{t\to 0} \int u(x, t)\eta(x)\,dx = \int \eta\, d\mu \quad \forall \eta \in C_0^\infty(\overline{\Omega}_0).$$

We let $\rho \in C_0^\infty(R^n)$, $\rho \ge 0$, $\int \rho = 1$ and for any g we define

$$g_\varepsilon = g * \rho_\varepsilon(x) = \int \rho_\varepsilon(x - y) g(y)\,dy, \quad \varepsilon > 0,$$

where $\rho_\varepsilon(y) = \rho(y/\varepsilon)/\varepsilon^n$. For any $r > 0$, $x_0 \in R^n$, let $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$. For any set $A \subset R^n$, we let χ_A be the characteristic function of the set A. We will also assume $m > 1$, $q > m + 1$, and let $u^{(q)}$ be the solution of (0.1) for the rest of the paper.

The plan of the paper is as follows. In section 1 we will state and prove the existence of solutions of (0.1). We will also prove a comparison theorem for solutions of (0.1) and obtain some bounds on $u^{(q)}$ by constructing explicit supersolutions to (0.1). In section 2 we will first prove a comparison lemma for solutions of (0.3). We then prove the main theorem under the assumption $f \in C_0^1(R^n)$ (Theorem 2.9). Finally we will prove the main theorem (Theorem 2.10) by an approximation argument.

1

We first state and prove an uniqueness theorem for solutions of (0.1).

Theorem 1.1. *If* $u_1^{(q)}, u_2^{(q)} \in L^\infty([0,1); L^1(R^n)) \cap L^\infty(R^n \times (0,1)) \cap C(R^n \times (0,1))$ *are the solutions of*

$$u_t = \Delta u^m - (u^q/q)_{x_1} \tag{1.1}$$

in $R^n \times (0,1)$ *with initial values* f_1 *and* $f_2 \in L^1(R^n) \cap L^\infty(R^n)$ *respectively,* $f_1, f_2 \geq 0$, *then there exists a constant* $C > 0$ *such that*

(i) $$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x,t)dx \leq e^{Ct} \int_{R^n} (f_1 - f_2)_+(x)dx,$$

(ii) $$\int_{R^n} |u_1^{(q)} - u_2^{(q)}|(x,t)dx \leq e^{Ct} \int_{R^n} |f_1 - f_2|(x)dx$$

for all $0 < t < 1$. *Hence* $u_1^{(q)} \leq u_2^{(q)}$ *if* $f_1 \leq f_2$. *In particular the solution of* (1.1) *in* $R^n \times (0,1)$ *with initial value in* $L^1(R^n) \cap L^\infty(R^n)$ *is unique in the class* $L^\infty([0,1); L^1(R^n)) \cap L^\infty(R^n \times (0,1)) \cap C(R^n \times (0,1))$.

Proof. The proof of the theorem is similar to the proof of Theorem 2.3 of [A]. By subtracting the equation for $u_1^{(q)}$ and $u_2^{(q)}$, we get

$$\begin{aligned} \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x,t)\eta(x,t)dx &= \int_{B_R(0)} (f_1 - f_2)(x)\eta(x,0)dx \\ &+ \int_0^t \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(\eta_t + A\Delta\eta + B\eta_{x_1})dxd\tau \\ &- \int_0^t \int_{\partial B_R(0)} (u_1^{(q)m} - u_2^{(q)m})\frac{\partial \eta}{\partial N} d\sigma d\tau \end{aligned} \tag{1.2}$$

for all $0 < t < 1$, $\eta \in C^\infty(\overline{B_R(0)} \times [0,t])$, $R > 0$, such that $\eta \equiv 0$ on $\partial B_R(0) \times [0,t]$ where

$$A = \begin{cases} \dfrac{u_1^{(q)m} - u_2^{(q)m}}{u_1^{(q)} - u_2^{(q)}} & \text{for } u_1^{(q)} \neq u_2^{(q)}, \\ m u_1^{(q)m-1} & \text{for } u_1^{(q)} = u_2^{(q)}, \end{cases}$$

$$B = \begin{cases} \dfrac{1}{q} \cdot \dfrac{u_1^{(q)q} - u_2^{(q)q}}{u_1^{(q)} - u_2^{(q)}} & \text{for } u_1^{(q)} \neq u_2^{(q)}, \\ u_1^{(q)q-1} & \text{for } u_1^{(q)} = u_2^{(q)}. \end{cases}$$

Since $u_1^{(q)}, u_2^{(q)} \in L^\infty(R^n \times (0,1))$, there exists a constant $C_1 > 0$ such that

$$\|u_1^{(q)}\|_{L^\infty(R^n)}, \|u_2^{(q)}\|_{L^\infty(R^n)} \leq C_1$$
$$\Rightarrow B^2/2A \leq \frac{1}{2m} C_1^{2q-m-1}, \; B/A \leq \frac{1}{m} C_1^{q-m}.$$

By an argument similar to section 4 of [A], there exists smooth functions $A_{i,R}$ and $B_{i,R}$ and constant $c_i > 0$ such that $c_i \le A_{i,R} \le mC_1^{m-1} + 1$, $0 \le B_{i,R} \le C_1^{q-1}+1$, $B_{i,R}^2/2A_{i,R} \le (C_1^{2q-m-1}/2m)+1 = C_2$, $B_{i,R}/A_{i,R} \le (C_1^{q-m}/m)+1 = C_3$, $(A_{i,R} - A)/A_{i,R}^{1/2} \to 0$ and $B_{i,R} - B \to 0$ in $L^2(B_R(0) \times (0, 1))$ as $i \to \infty$ for all $R > 0$.

For any $R_0 > 2$, $R > R_0 + 1$, $\lambda > C_2$, $\theta \in C_0^\infty(B_{R_0}(0))$, $0 \le \theta \le 1$, let $\eta_{i,R}$ be the solution of

$$\begin{cases} \eta_s + A_{i,R}\Delta\eta + B_{i,R}\eta_{x_1} - \lambda\eta = 0 & \text{for } (x, s) \in B_R(0) \times (0, t), \\ \eta(x, s) = 0 & \text{for } (x, s) \in \partial B_R(0) \times (0, t], \\ \eta(x, t) = \theta(x) & \text{for } x \in B_R(0). \end{cases}$$

Since $0 \le \theta \le 1$, by the maximum principle $0 \le \eta_{i,R} \le 1$. By Lemma 4.1 of [A], we have

$$\begin{aligned} &\int_0^t \int_{B_R(0)} A_{i,R}(\Delta\eta_{i,R})^2 dx d\tau + 2(\lambda - C_2) \int_0^t \int_{B_R(0)} |\nabla\eta_{i,R}|^2 dx d\tau \\ &\quad \le \int_{B_R(0)} |\nabla\theta|^2 dx. \end{aligned} \tag{1.3}$$

By the same argument as the proof of Theorem 2.1 (ii) of [PV], we see that for any $\beta > 0$, the function

$$g(x, s) = e^{h(s)} \left(\frac{1 + R_0^2}{1 + |x|^2} \right)^\beta$$

where $h(s) = C'(t - s)$, $C' = 4\beta(\beta + 1)(mC_1^{m-1} + 1) + \beta(C_1^{q-1} + 1)$, satisfies

$$\begin{cases} g_s + A_{i,R}\Delta g + B_{i,R} g_{x_1} - \lambda g < 0, & \text{for } (x, s) \in B_R(0) \times (0, t), \\ g(x, s) \ge \eta_{i,R}(x, s), & \text{for } (x, s) \in B_R(0) \times \{t\} \cup \partial B_R(0) \times (0, t]. \end{cases}$$

Hence by the maximum principle [LSU], $g \ge \eta_{i,R}$ in $B_R(0) \times (0, t)$. We next consider the function

$$g^*(x, s) = a e^{h(s)} \Gamma(|x|), \quad R - \alpha \le r \le R, 0 \le s \le t,$$

where $\alpha = 1/2(C_3 + n - 1)$, $\Gamma(r) = (R - r) - C_3(R - r)^2$ and

$$a = (1 + R_0^2)^\beta / \{\Gamma(R - \alpha)(1 + (R - \alpha)^2)^\beta\}.$$

Then $g^* \ge 0$, $g_s^* = h'(s) g^* \le 0$ and

$$\begin{aligned} &\Delta g^* + (B_{i,R}/A_{i,R}) g_{x_1} \\ &\quad \le a e^{h(s)} \left(\Gamma''(r) + \frac{n-1}{r}\Gamma'(r) + C_3|\Gamma'(r)| \right) \\ &\quad \le a e^{h(s)} \left(-2C_3 + \frac{n-1}{r}(-1 + 2C_3(R - r)) + C_3(1 + 2C_3(R - r)) \right) \\ &\quad \le a C_3 e^{h(s)} (-1 + 2(C_3 + n - 1)(R - r)) \\ &\quad \le 0 \end{aligned}$$

for all $R - \alpha < r < R$, $0 < s < t$ since $R - \alpha \ge R_0 \ge 2$. Hence g^* satisfies

$$g_s^* + A_{i,R}\Delta g^* + B_{i,R} g_{x_1}^* - \lambda g^* < 0, \quad \text{for } (x, s) \in B_R(0) \setminus B_{R-\alpha}(0) \times (0, t)$$

with $g^*(x, s) \ge \eta_{i,R}(x, s)$ for all

$$(x, s) \in B_R(0) \setminus B_{R-\alpha}(0) \times \{t\} \cup (\partial B_R(0) \cup \partial B_{R-\alpha}(0)) \times (0, t].$$

By the maximum principle, $0 \le \eta_{i,R} \le g^*$ in $B_R(0) \setminus B_{R-\alpha}(0) \times (0, t)$. Since $g^* \equiv \eta_{i,R} \equiv 0$ on $\partial B_R(0) \times [0, t]$,

$$\|\partial \eta_{i,R}/\partial N\|_{L^\infty(\partial B_R(0)\times(0,t))} \le \|\partial g^*/\partial N\|_{L^\infty(\partial B_R(0)\times(0,t))} \le CR^{-2\beta}. \tag{1.4}$$

Putting $\eta = \eta_{i,R}$ in (1.2), we get by (1.3) and (1.4),

(1.5)

$$\begin{aligned}
&\int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx \\
&\quad = \int_{B_R(0)} (f_1 - f_2)(x)\eta_{i,R}(x, 0)dx + \int_0^t \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})[(A - A_{i,R})\Delta\eta_{i,R} \\
&\qquad + (B - B_{i,R})(\eta_{i,R})_{x_1} + \lambda\eta_{i,R}]dxd\tau \\
&\qquad - \int_0^t \int_{\partial B_R(0)} (u_1^{(q)m} - u_2^{(q)m})\frac{\partial \eta_{i,R}}{\partial N} d\sigma d\tau \\
&\quad \le \int_{R^n} (f_1 - f_2)_+ dx + 2C_1\|(A_{i,R} - A)/A_{i,R}^{1/2}\|_{L^2(B_R(0)\times(0,t))}\|\nabla\theta\|_{L^2(B_R(0))} \\
&\qquad + (2C_1/(2(\lambda - C_2))^{1/2})\|B_{i,R} - B\|_{L^2(B_R(0)\times(0,t))}\|\nabla\theta\|_{L^2(B_R(0))} \\
&\qquad + \lambda \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau + CR^{n-1-2\beta}t
\end{aligned}$$

for all $\theta \in C_0^\infty(B_{R_0}(0))$, $0 \le \theta \le 1$, $0 < t < 1$. Choose now $\beta = n/2$ and let first $i \to \infty$ and then $R \to \infty$, $\lambda \to C_2$ in (1.5), we get

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx \le \int_{R^n} (f_1 - f_2)_+ dx + C_2 \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau$$

for all $\theta \in C_0^\infty(B_{R_0}(0))$, $0 \le \theta \le 1$, $R_0 > 2$. Putting $\theta = \chi_{\{u_1^{(q)} \ge u_2^{(q)}\} \cap B_{R_0 - 1}(0)} * \rho_\varepsilon$ into the above inequality and letting first $\varepsilon \to 0$ and then $R_0 \to \infty$, we get

$$\begin{aligned}
&\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)dx \\
&\qquad \le \int_{R^n} (f_1 - f_2)_+ dx + C_2 \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau \quad \forall 0 < t < 1.
\end{aligned}$$

(i) then follows from the Gronwall's inequality. Similarly,

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_-(x, t)dx \le e^{C_2 t} \int_{R^n} (f_1 - f_2)_- dx \quad \forall 0 < t < 1.$$

By combining the above inequality with (i), we get (ii).

Corollary 1.2. *If $u_1^{(q)}$ is a subsolution and $u_2^{(q)}$ is a supersolution of (1.1) in $Q = D \times (0, 1)$ where $D = (-\infty, R_0] \times R^{n-1}$ for some $R_0 \in R$ (or $D = [R_0, R_1] \times R^{n-1}$ for some R_0, $R_1 \in R$, $R_0 < R_1$) with $u_1^{(q)}$, $u_2^{(q)} \in L^\infty([0, 1); L^1(D)) \cap L^\infty(D \times (0, 1)) \cap C(D \times (0, 1))$ with initial values $u_1^{(q)}(x, 0)$, $u_2^{(q)}(x, 0)$ and boundary values satisfying*

$$u_1^{(q)}(x, t) \le u_2^{(q)}(x, t) \quad \forall (x, t) \in \partial_p Q$$

where $\partial_p Q = \{R_0\}\times R^{n-1}\times(0,1)\cup(-\infty, R_0]\times R^{n-1}\times\{0\}$ (*respectively* $\partial_p Q = \{R_0, R_1\}\times R^{n-1}\times(0,1)\cup[R_0, R_1]\times R^{n-1}\times\{0\}$), *then*

$$u_1^{(q)}(x,t)\le u_2^{(q)}(x,t)\quad \forall (x,t)\in Q$$

Proof. The proof is the same as the proof of Theorem 1.1.

Theorem 1.3. *The equation*

$$(1.6)\qquad \begin{cases} u_t = \Delta u^m - (u^q/q)_{x_1},\ u\ge 0, & (x,t)\in R^n\times(0,1),\\ u(x,0)=f(x)\ge 0, & f\in L^1(R^n)\cap L^\infty(R^n),\end{cases}$$

has a unique solution $u^{(q)}\in L^\infty([0,1);\ L^1(R^n))\cap L^\infty(R^n\times(0,1))\cap C(R^n\times(0,1))$ *with*

$$(1.7)\qquad \text{(i)}\qquad \int u^{(q)}(x,t)dx = \int f dx\quad \forall 0<t<1,$$

$$(1.8)\qquad \text{(ii)}\qquad \|u^{(q)}\|_{L^\infty(R^n\times(0,1))}\le \|f\|_{L^\infty(R^n)}.$$

Proof. The proof is similar to that of [ERV] and [DK]. Let $\psi\in C_0^\infty(R^n)$, $0\le\psi\le 1$, be such that $\psi(x)\equiv 1$ for all $|x|\le 1/2$ and $\psi\equiv 0$ for all $|x|\ge 1$. For any $\varepsilon>0$, $0<\varepsilon<1$, $R>0$, let $f_{\varepsilon,R}(x) = f*\rho_\varepsilon(x)\cdot\psi(x/R)+\varepsilon$ and let $a_\varepsilon(s)$, $b_\varepsilon(s)\in C^\infty(R)$ be such that $a_\varepsilon'(s)$, $b_\varepsilon'(s)\ge 0$,

$$a_\varepsilon(s) = \begin{cases} m(\|f\|_{L^\infty(R^n)}+2)^{m-1} & \text{for } s\ge \|f\|_{L^\infty(R^n)}+2,\\ ms^{m-1} & \text{for } \varepsilon\le s\le \|f\|_{L^\infty(R^n)}+1,\\ m(\varepsilon/2)^{m-1} & \text{for } s\le\varepsilon/2,\end{cases}$$

$$b_\varepsilon(s) = \begin{cases} (\|f\|_{L^\infty(R^n)}+2)^{q-1} & \text{for } s\ge \|f\|_{L^\infty(R^n)}+2,\\ s^{q-1} & \text{for } \varepsilon\le s\le \|f\|_{L^\infty(R^n)}+1,\\ (\varepsilon/2)^{q-1} & \text{for } s\le\varepsilon/2.\end{cases}$$

By standard parabolic theory [LSU], there exists a unique solution $u_{\varepsilon,R}^{(q)}$ to the equation

$$(1.9)\qquad \begin{cases} u_t = \operatorname{div}(a_\varepsilon(u)\nabla u) - b_\varepsilon(u)u_{x_1}, & \text{for } (x,t)\in B_R(0)\times(0,1),\\ u(x,t)=\varepsilon & \text{for } (x,t)\in\partial B_R(0)\times(0,1),\\ u(x,0)=f_{\varepsilon,R}(x), & \text{for } x\in B_R(0).\end{cases}$$

Since $\varepsilon\le f_{\varepsilon,R}\le\|f\|_{L^\infty(R^n)}+\varepsilon$, by the maximum principle,

$$(1.10)\qquad \varepsilon\le u_{\varepsilon,R}^{(q)}\le \|f\|_{L^\infty(R^n)}+\varepsilon.$$

Hence $a_\varepsilon(u_{\varepsilon,R}^{(q)}) = mu_{\varepsilon,R}^{(q)m-1}$, $b_\varepsilon(u_{\varepsilon,R}^{(q)}) = u_{\varepsilon,R}^{(q)q-1}$. Since (1.3) is a nondegenerate parabolic equation, by Schauder's estimate [LSU], $u_{\varepsilon,R}^{(q)}\in C^\infty(B_R(0)\times[0,1))$. Thus $u_{\varepsilon,R}^{(q)}$ satisfies (1.1) in $B_R(0)\times(0,1)$. Since $u_{\varepsilon,R}^{(q)}$ is uniformly bounded by $\|f\|_{L^\infty(R^n)}+1$, by the result of P. Sacks [S1], $\{u_{\varepsilon,R}^{(q)}\}_{R>0}$ has a convergent subsequent $\{u_{\varepsilon,R_j}^{(q)}\}_{j=1}^\infty$, $R_j\to\infty$ as $j\to 0$, such that $\{u_{\varepsilon,R_j}^{(q)}\}_{j=1}^\infty$ converges uniformly on compact subsets of $R^n\times(0,1)$. Let $u_\varepsilon^{(q)} = \lim_{j\to\infty} u_{\varepsilon,R_j}^{(q)}$. Then $u_\varepsilon^{(q)}\in C(R^n\times(0,1))$ and

$$(1.11)\qquad \varepsilon\le u_\varepsilon^{(q)}\le\|f\|_{L^\infty(R^n)}+\varepsilon.$$

Putting $u = u^{(q)}_{\varepsilon, R_j}$ in (1.1) and letting $j \to 0$, we see that $u^{(q)}_\varepsilon$ satisfies (1.1) in $R^n \times (0, 1)$ with $u^{(q)}_\varepsilon(x, 0) = f * \rho_\varepsilon(x) + \varepsilon$. Thus $u^{(q)}_\varepsilon \in C^\infty(R^n \times (0, 1))$ by (1.11) and Schauder's estimates. Since $\|u^{(q)}_\varepsilon\|_{L^\infty(R^n \times (0,1))} \le \|f\|_{L^\infty(R^n)} + \varepsilon$, by [S1], $\{u^{(q)}_\varepsilon\}_{\varepsilon > 0}$ has a convergent subsequent $\{u^{(q)}_{\varepsilon_i}\}_{i=1}^\infty$, $\varepsilon_i \to 0$ as $i \to 0$, such that $\{u^{(q)}_{\varepsilon_i}\}_{i=1}^\infty$ converges uniformly on compact subsets of $R^n \times (0, 1)$. Let $u^{(q)} = \lim_{i \to \infty} u^{(q)}_{\varepsilon_i}$. Then $u^{(q)} \in C(R^n \times (0, 1))$.

Putting $u = u^{(q)}_{\varepsilon_i}$ in (1.1) and letting $i \to 0$, we see that $u^{(q)}$ satisfies (1.1) in $R^n \times (0, 1)$. Moreover,

$$\begin{aligned}
&\left| \int_{R^n} u^{(q)}_{\varepsilon, R_j}(x, t)\eta(x)dx - \int_{R^n} f_{\varepsilon, R_j} \eta dx \right| \\
&= \left| \int_0^t \int_{R^n} (u^{(q)}_{\varepsilon, R_j})_t(x, \tau)\eta(x)dxd\tau \right| \\
&= \left| \int_0^t \int_{R^n} \left[\Delta u^{(q)m}_{\varepsilon, R_j} - \left(\frac{u^{(q)q}_{\varepsilon, R_j}}{q} \right)_{x_1} \right] \eta dxd\tau \right| \qquad (1.12) \\
&= \left| \int_0^t \int_{R^n} \left(u^{(q)m}_{\varepsilon, R_j} \Delta\eta + \frac{u^{(q)q}_{\varepsilon, R_j}}{q} \eta_{x_1} \right) dxd\tau \right| \\
&\le (\|f\|_{L^\infty(R^n)} + 1)^m \|\Delta\eta\|_{L^1(R^n)} t + \frac{(\|f\|_{L^\infty(R^n)} + 1)^q}{q} \|\eta_{x_1}\|_{L^1(R^n)} t
\end{aligned}$$

for all $\eta \in C_0^\infty(R^n)$. Letting first $j \to 0$ and then $\varepsilon = \varepsilon_i \to 0$, $t \to 0$, we get

$$\lim_{t \to 0} \int_{R^n} u^{(q)}(x, t)\eta(x)dx = \int_{R^n} f\eta dx \quad \forall \eta \in C_0^\infty(R^n).$$

Hence $u^{(q)}$ has initial trace f and $\|u^{(q)}\|_{L^\infty(R^n \times (0,1))} \le \|f\|_{L^\infty(R^n)}$ by (1.11). On the other hand, since $u^{(q)}_\varepsilon$ satisfies (1.1) in $R^n \times (0, 1)$,

$$\begin{aligned}
\int_{B_R(0)} u^{(q)}_\varepsilon(x, t)\eta(x, t)dx &= \int_{B_R(0)} (f * \rho_\varepsilon(x) + \varepsilon)\eta(x, 0)dx \\
&\quad + \int_0^t \int_{B_R(0)} u^{(q)}_\varepsilon (\eta_t + A_\varepsilon \Delta\eta + B_\varepsilon \eta_{x_1}) dxd\tau \qquad (1.13) \\
&\quad - \int_0^t \int_{\partial B_R(0)} u^{(q)m}_\varepsilon \frac{\partial \eta}{\partial N} d\sigma d\tau
\end{aligned}$$

for all $0 < t < 1$, $\eta \in C^\infty(\overline{B_R(0)} \times [0, t])$, $R > 0$ such that $\eta \equiv 0$ on $\partial B_R(0) \times [0, t]$ where $A_\varepsilon = u^{(q)m-1}_\varepsilon$, $B_\varepsilon = u^{(q)q-1}_\varepsilon / q$.

For any $R_0 > 2$, $R > R_0 + 1$, $\theta \in C_0^\infty(B_{R_0}(0))$, $0 \le \theta \le 1$, $\theta \equiv 1$ for $|x| \le R_0 - 1$, let $\eta_{\varepsilon, R}$ be the solution of

$$\begin{cases}
\eta_s + A_\varepsilon \Delta\eta + B_\varepsilon \eta_{x_1} = 0 & \text{for } (x, s) \in B_R(0) \times (0, t), \\
\eta(x, s) = 0 & \text{for } (x, s) \in \partial B_R(0) \times (0, t], \\
\eta(x, t) = \theta(x) & \text{for } x \in B_R(0).
\end{cases}$$

By an argument similar to the proof of Theorem 1.1, we have $0 \le \eta_{\varepsilon, R} \le 1$,

$$\eta_{\varepsilon, R}(x, s) \le e^{h(s)} \left(\frac{1 + R_0^2}{1 + |x|^2} \right)^n \quad \forall 0 \le s \le t,$$

where $h(s) = C'(t-s)$, $C' = 4n(n+1)(b_1^{m-1}+1)+n(b_1^{q-1}+1)$, $b_1 = \|f\|_{L^\infty(R^n)}+1$, and

$$\|\partial\eta_{\varepsilon,R}/\partial N\|_{L^\infty(\partial B_R(0)\times(0,t))} \le CR^{-2n}$$

for some constant $C > 0$ depending only on R_0 and b_1. Putting $\eta = \eta_{\varepsilon,R}$ into (1.13), we get

$$\int_{B_R(0)} u_\varepsilon^{(q)}\theta(x)dx \le \int f dx + C'_{R_0}R^{-n-1} + \varepsilon C_{R_0} \tag{1.14}$$

for some constant C_{R_0}, $C'_{R_0} > 0$ depending only on R_0 and b_1. Letting $R \to \infty$, $\varepsilon = \varepsilon_i \to 0$,

$$\int_{|x|\le R_0-1} u^{(q)}(x,t)dx \le \int_{R^n} u_\varepsilon^{(q)}(x,t)\theta(x)dx \le \int f dx$$

for all $0 < t < 1$. Letting $R_0 \to \infty$,

$$\int_{R^n} u^{(q)}(x,t)dx \le \int f dx \quad \forall 0 < t < 1. \tag{1.15}$$

Hence $u^{(q)} \in L^\infty([0,1); L^1(R^n))\cap L^\infty(R^n\times(0,1))\cap C(R^n\times(0,1))$ and satisfies (1.8). It remains to show (1.7). Since

$$\begin{aligned}(1.15) \Rightarrow \quad & \int_0^1\int_{R^n} u^{(q)}(x,\tau)dxd\tau \le \int_{R^n} f dx \\ \Rightarrow \quad & \int_0^1\int_{R/2\le|x|\le R} u^{(q)}(x,\tau)dxd\tau \to 0 \quad \text{as } R\to\infty,\end{aligned}$$

putting $\eta(x) = \psi(x/R)$, $R > 0$, in (1.12), we have

$$\begin{aligned}&\left|\int_{R^n} u_{\varepsilon,R_j}^{(q)}(x,t)\psi(x/R)dx - \int_{R^n} f_{\varepsilon,R_j}(x)\psi(x/R)dx\right| \\ &\quad\le \frac{(\|f\|_{L^\infty(R^n)}+1)^{m-1}}{R^2}\|\Delta\psi\|_{L^\infty(R^n)}\int_0^t\int_{R/2\le|x|\le R} u_{\varepsilon,R_j}^{(q)}(x,\tau)dxd\tau \\ &\quad\quad + \frac{(\|f\|_{L^\infty(R^n)}+1)^{q-1}}{qR}\|\psi_{x_1}\|_{L^\infty(R^n)}\int_0^t\int_{R/2\le|x|\le R} u_{\varepsilon,R_j}^{(q)}(x,\tau)dxd\tau.\end{aligned}$$

By letting first $j\to\infty$ and then $\varepsilon = \varepsilon_i \to 0$, $R\to\infty$, in the above inequality, we get (1.7). Since uniqueness of solution of (1.6) follows from Theorem 1.1. This completes the proof of the theorem.

Theorem 1.4. *Let* $u_1^{(q)}$, $u_2^{(q)}$, f_1, f_2 *be as in Theorem* 1.1. *Then*

$$\int_{R^n} |u_1^{(q)} - u_2^{(q)}|(x,t)dx \le \int_{R^n} |f_1 - f_2|dx \quad \forall 0 < t < 1.$$

Proof. By Theorem 1.1 and the proof of Theorem 1.3, there exist solutions $u_{1,\varepsilon}^{(q)}$, $u_{2,\varepsilon}^{(q)} \in C^\infty(R^n\times(0,1))\cap L^\infty(R^n\times(0,1))$, $0 < \varepsilon < 1$, of (1.6) with initial values $u_{1,\varepsilon}^{(q)}(x,0) = f_1 * \rho_\varepsilon + \varepsilon$, $u_{2,\varepsilon}^{(q)}(x,0) = f_2 * \rho_\varepsilon + \varepsilon$ respectively such that $u_{1,\varepsilon}^{(q)}$ and $u_{2,\varepsilon}^{(q)}$ converges uniformly to $u_1^{(q)}$ and $u_2^{(q)}$ respectively on compact subsets of $R^n\times(0,1)$ as $\varepsilon\to 0$.

By a proof similar to the proof of (1.14), we have

$$\int_{B_R(0)} (u_{1,\varepsilon}^{(q)} - u_{2,\varepsilon}^{(q)})(x, t)\theta(x)dx \le \int_{R^n} (f_1 * \rho_\varepsilon - f_2 * \rho_\varepsilon)_+ dx + C'_{R_0} R^{-1-n} + \varepsilon C_{R_0}$$

for all $\theta \in C_0^\infty(B_{R_0}(0))$, $R_0 > 2$, $R > R_0 + 1$, $0 < t < 1$ where C_{R_0} and $C'_{R_0} > 0$ are constants depending only on R_0, $\|u_1^{(q)}\|_{L^\infty(R^n)}$ and $\|u_2^{(q)}\|_{L^\infty(R^n)}$. Letting $R \to \infty$, $\varepsilon \to 0$, we get

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx \le \int_{R^n} (f_1 - f_2)_+ dx$$

for all $\theta \in C_0^\infty(B_{R_0}(0))$, $R_0 > 2$, $0 < t < 1$. Putting $\theta = \chi_{\{u_1^{(q)} \ge u_2^{(q)}\}} * \rho_\varepsilon$ and letting first $\varepsilon \to 0$ and then $R_0 \to \infty$,

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)dx \le \int_{R^n} (f_1 - f_2)_+ dx \quad \forall 0 < t < 1.$$

Similarly,

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_-(x, t)dx \le \int_{R^n} (f_1 - f_2)_- dx \quad \forall 0 < t < 1.$$

Combining the above two inequalities the theorem follows.

Lemma 1.5. *If $f \in C_0^1(R^n)$ and $f_\varepsilon = f + \varepsilon$, $0 < \varepsilon < 1$, then (1.1) has a unique solution $u_\varepsilon^{(q)} \in C^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1))$ in $R^n \times (0, 1)$ with $u_\varepsilon^{(q)}(x, 0) = f_\varepsilon(x)$ such that $u_\varepsilon^{(q)}$ converges uniformly on compact subsets of $R^n \times (0, 1)$ to the solution $u^{(q)}$ of (1.6) with $u^{(q)}(x, 0) = f(x)$ as $\varepsilon \to 0$. Moreover*

$$\|u_{\varepsilon, x_k}^{(q)}\|_{L^\infty(R^n)} \le \|f_{x_k}\|_{L^\infty(R^n)} \quad \forall k = 1, 2, \dots, n.$$

Proof. By Theorem 1.4 and an argument similar to the proof of Theorem 1.3, for any $0 < \varepsilon < 1$ there exists a unique solution $u_\varepsilon^{(q)} \in C^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1))$ to (1.1) in $R^n \times (0, 1)$ with $u_\varepsilon^{(q)}(x, 0) = f(x) + \varepsilon$ and

$$\varepsilon \le u_\varepsilon^{(q)} \le \|f\|_{L^\infty(R^n)} + \varepsilon \tag{1.16}$$

such that $u_\varepsilon^{(q)}$ converges uniformly on compact subsets of $R^n \times (0, 1)$ to the solution $u^{(q)}$ of (1.6) with $u^{(q)}(x, 0) = f(x)$ as $\varepsilon \to 0$.

Since $u_\varepsilon^{(q)} \in C^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1))$, differentiating (1.1) with respect to x_k and writing $z = u_{\varepsilon, x_k}^{(q)}$, we get

$$\begin{cases} z_t = \Delta(m u_\varepsilon^{(q)m-1} z) + (u_\varepsilon^{(q)q-1} z)_{x_1}, & (x, t) \in R^n \times (0, 1), \\ z(x, 0) = f_{x_k}(x), & x \in R^n, \end{cases}$$

for all $k = 1, 2, \dots, n$. Since the above equation is nondegenerate by (1.16), by the maximum principle,

$$\|z\|_{L^\infty(R^n)} \le \|f_{x_k}\|_{L^\infty(R^n)} \quad \forall k = 1, 2, \dots, n$$

and the lemma follows.

Lemma 1.6. *Let $0 \le f \le M$ with $\operatorname{supp} f \subset B_{R_1}(0)$ for some $R_1 > 0$. Then there exists $R' > 0$ depending only on m, R_1, M and is independent of $q > m + 1$ such that*

$$u^{(q)}(x, t) = 0 \quad \forall x = (x_1, \dots, x_n) \in R^n, x_1 \le -R', 0 \le t < 1, q > m + 1,$$

and

$$0 \le u^{(q)}(x, t) \le \left(\frac{x_1 + R' + 1}{t + (1/M^{q-1})}\right)^{1/q-1} \le \left(\frac{x_1 + R' + 1}{t}\right)^{1/q-1}$$

for all $x = (x_1, \dots, x_n) \in R^n$, $x_1 \ge -R'$, $0 < t < 1$, $q > m+1$.

Proof. Let

$$w(x_1, t) = \frac{1}{(t+t_0)^{1/m+1}} \left(a^2 - C_1 \left(\frac{x_1}{(t+t_0)^{1/m+1}}\right)^2\right)_+^{1/m-1}, \quad x_1 \in R, t \ge 0,$$

be the Barenblatt solution for the porous medium equation $w_t = (w^m)_{x_1x_1}$ ([B], [HP]) where $C_1 = \frac{m-1}{2m}\left(\frac{1}{(m+1)}\right)$, $t_0 = \min\left(1, \left(\frac{4C_1R_1^2}{2^{m-1/m+1}M^{m-1}}\right)^{(m+1)/2}\right)$ and

$$a = (C_1(2R_1/t_0^{1/m+1})^2 + (Mt_0^{1/m+1})^{m-1})^{1/2}.$$

Then w is a supersolution of (1.1) in $(-\infty, 0] \times R^{n-1} \times (0, 1)$ with

$$u^{(q)}(x + x_0) = f(x + x_0) \le M \le w(x_1, 0)$$

for all $x = (x_1, \dots, x_n) \in R^n$, $x_1 \le 0$ where $x_0 = (R_1, 0, \dots, 0)$ and

$$\begin{aligned} w(0, t) &\ge \frac{1}{(1+t_0)^{1/m+1}} a^{2/m-1} \\ &\ge \frac{1}{2^{1/m+1}}\left(C_1\left(\frac{2R_1}{t_0^{1/m+1}}\right)^2\right)^{1/m-1} \\ &\ge \frac{1}{2^{1/m+1}}\left(\frac{4R_1^2C_1}{4C_1R_1^2/(2^{1/m+1}M)^{m-1}}\right)^{1/m-1} \\ &= M \ge u(x_0, t) \end{aligned}$$

for all $0 < t < 1$ by (1.8). Hence by applying the maximum principle (Corollary 1.2) to the functions $u^{(q)}(\cdot + x_0, \cdot)$ and w in the region $(-\infty, 0] \times R^{n-1} \times (0, 1)$, we get

$$u^{(q)}(x + x_0, t) \le w(x_1, t) \quad \forall x = (x_1, \dots, x_n) \in R^n, x_1 \le 0, 0 \le t < 1.$$

Now for each $0 < t < 1$, supp $w(x_1, t) \subset B_{R_t}(0)$ where

$$R_t = \frac{a}{C_1^{1/2}}(t + t_0)^{1/m+1} \le \frac{2a}{C_1^{1/2}} \quad (= R_2 \text{ say }).$$

Hence

$$\begin{aligned} u^{(q)}(x + x_0, t) \le w(x_1, t) = 0 \quad \forall x = (x_1, \dots, x_n) \in R^n, \\ x_1 \le -R_2, 0 \le t < 1, \end{aligned}$$

$$\begin{aligned} \Rightarrow u^{(q)}(x, t) = 0 \quad \forall x = (x_1, \dots, x_n) \in R^n, x_1 \le -R', 0 \le t < 1, \\ q > m+1, \end{aligned} \tag{1.17}$$

where $R' = \max(R_2 - R_1, 0) \ge 0$.

We next observe that

$$\widetilde{w}(x_1, t) = \left(\frac{x_1 + R' + 1}{t + (1/M^{q-1})}\right)^{1/q-1}, \quad q > m+1,$$

is a supersolution of (1.1) in $[-R', R_3] \times R^{n-1} \times (0, \infty)$ with

$$\begin{cases} u^{(q)}(x, 0) \le M \le \widetilde{w}(x_1, 0) & \text{for } x = (x_1, \dots, x_n) \in R^n, \ -R' \le x_1 \le R_3, \\ u^{(q)}(x, t) \le M \le \widetilde{w}(x_1, t) & \text{for } x = (x_1, \dots, x_n) \in R^n, \\ & x_1 = -R' \text{ or } x_1 = R_3, \ 0 \le t < 1, \end{cases}$$

for all $R_3 > \max(2M^{q-1} - R' + 1, 0)$ by (1.17). Hence by applying Corollary 1.2 to the function $u^{(q)}$ and $\widetilde{w}$ in the region $[-R', R_3] \times R^{n-1} \times (0, 1)$, we get

$$u^{(q)}(x, t) \le \widetilde{w}(x_1, t)$$

for all $x = (x_1, \dots, x_n) \in [-R', R_3] \times R^{n-1}$, $0 \le t < 1$, $q > m+1$, $R_3 > \max(2M^{q-1} - R' + 1, 0)$. By letting $R_3 \to \infty$, the lemma follows.

Lemma 1.7. *Suppose f is as in Lemma 1.6. Let $\Omega \subset R^n$ be a bounded open set with $\partial\Omega \in C^2$ and $\eta \in C^\infty(R^n \times (0, 1))$. Then*

$$\int_{\tau_1}^{\tau_2} \int_\Omega \frac{u^{(q)q}}{q} \cdot \eta \, dx\, dt \to 0 \quad \text{as } q \to \infty$$

for any $0 < \tau_1 \le \tau_2 < 1$.

Proof. By Lemma 1.6, there exists a constant $R' > 0$ such that

$$u^{(q)}(x, t) \le \left(\frac{|x_1| + R' + 1}{t}\right)^{1/q-1} \quad \forall x = (x_1, \dots, x_n) \in R^n, \ 0 < t < 1, \ q > m+1,$$

and by Theorem 1.3 $\|u^{(q)}\|_{L^\infty} \le \|f\|_{L^\infty}$ for all $q > m+1$. Hence

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_\Omega \frac{u^{(q)q}}{q} \cdot \eta \, dx\, dt &= \int_{\tau_1}^{\tau_2} \int_\Omega \frac{u^{(q)q-2} u^{(q)2}}{q} \cdot \eta \, dx\, dt \\ &\le \|\eta\|_{L^\infty} \|f\|_{L^\infty}^2 \int_{\tau_1}^{\tau_2} \int_\Omega \frac{1}{q} \left(\frac{|x_1| + R' + 1}{t}\right)^{q-2/q-1} dx\, dt \\ &\le \frac{\|\eta\|_{L^\infty} \|f\|_{L^\infty}^2 |\Omega|}{q} \left(\frac{R'' + R' + 1}{\tau_1}\right)^{q-2/q-1} (\tau_2 - \tau_1) \to 0 \end{aligned}$$

as $q \to \infty$ where $R'' = \sup\{|x_1| : x = (x_1, \dots, x_n) \in \Omega\} < \infty$.

Lemma 1.8. *Let $f \in C_0(R^n)$ and let $p^{(q)}(x, t) = \int_0^t \frac{u^{(q)q}(x, \tau)}{q} d\tau$. Then $\{p^{(q)}\}_{q > m+1}$ is uniformly bounded on compact subsets of $R^n \times [0, 1)$. For any sequence $\{p^{(q_i)}\}_{i=1}^\infty$, $q_i \to \infty$ as $i \to \infty$, of $\{p^{(q)}\}_{q>m+1}$, there exists a subsequence $\{p^{(q_i')}\}_{i=1}^\infty$ of $\{p^{(q_i)}\}_{i=1}^\infty$, a sequence of functions $\{p_j\}_{j=1}^\infty \subset L^\infty_{\text{loc}}(R^n)$, $\widetilde{g} \in L^\infty_{\text{loc}}(R^n)$, $p_j, \widetilde{g} \ge 0$, and a sequence $\{\varepsilon_j\}_{j=1}^\infty \subset R$, $\varepsilon_j \to 0$ as $j \to \infty$, such that*

(1.18)

$$\begin{cases} p^{(q_i')}(\cdot, \varepsilon_j) \to p_j(\cdot) & \text{weakly in } (L^\infty(K))^* \text{ as } i \to \infty, \quad \forall j = 1, 2, \dots, \\ p_j(\cdot) \to \widetilde{g}(\cdot) & \text{weakly in } (L^\infty(K))^* \text{ as } j \to \infty \end{cases}$$

for any compact subset $K \subset R^n$.

Proof. By Theorem 1.3, $\|u^{(q)}\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)}$ for all $q > m+1$ and by Lemma 1.6 there exists $R' > 0$ such that

$$0 \le u^{(q)}(x, \tau) \le \left(\frac{|x_1| + R' + 1}{\tau}\right)^{1/(q-1)} \quad \forall x = (x_1, x_2, \dots, x_n) \in R^n,$$
$$0 < \tau < 1, \; q > m+1.$$

Hence

$$\begin{aligned} 0 \le p^{(q)}(x, t) &= \int_0^t \frac{u^{(q)q-2} u^{(q)2}}{q} d\tau \\ &\le \frac{\|f\|^2_{L^\infty(R^n)}}{q} \int_0^t \left(\frac{|x_1| + R' + 1}{\tau}\right)^{q-2/q-1} d\tau \\ &\le \frac{q-1}{q} \|f\|^2_{L^\infty(R^n)} (|x_1| + R' + 1)^{q-2/q-1} t^{1/q-1} \\ &\le \|f\|^2_{L^\infty(R^n)} (|x_1| + R' + 1) \end{aligned}$$

for all $x = (x_1, x_2, \dots, x_n) \in R^n$, $0 < t < 1$, $q > m+1$. Thus $\{p^{(q)}\}_{q>m+1}$ is uniformly bounded on compact subsets of $R^n \times [0, 1)$. So any sequence $\{p^{(q_i)}\}_{i=1}^\infty$ of $\{p^{(q)}\}_{q>m+1}$ will have a subsequence $\{p^{(q_{1,i})}\}_{i=1}^\infty$ such that $\{p^{(q_{1,i})}(\cdot, 1/2)\}_{i=1}^\infty$ converges weakly in $(L^\infty(K))^*$ for any compact subset $K \subset R^n$.

Let $p_1(\cdot) = \lim_{i\to\infty} p^{(q_{1,i})}(\cdot, 1/2)$. Then $\{p^{(q_{1,i})}(\cdot, 1/2)\}$ has a subsequence $\{p^{(q'_{1,i})}(\cdot, 1/2)\}$ such that $p^{(q'_{1,i})}(x, 1/2) \to p_1(x)$ a.e. $x \in R^n$ as $i \to \infty$. Without loss of generality we may assume $p^{(q_{1,i})}(x, 1/2) \to p_1(x)$ a.e. $x \in R^n$ as $i \to \infty$. We may also assume that $q_1 < q_{1,1}$. Since $\{p^{(q_{1,i})}(\cdot, 1/3)\}_{i=1}^\infty$ is uniformly bounded on compact subsets of R^n, $\{p^{(q_{1,i})}(\cdot, 1/3)\}_{i=1}^\infty$ has a subsequence $\{p^{(q_{2,i})}(\cdot, 1/3)\}_{i=1}^\infty$ converging weakly in $(L^\infty(K))^*$ for any compact set $K \subset R^n$. Let $p_2(\cdot) = \lim_{i\to\infty} p^{(q_{2,i})}(\cdot, 1/3)$. We may assume without loss of generality that $p^{(q_{2,i})}(x, 1/3) \to p_2(x)$ a.e. $x \in R^n$ as $i \to \infty$ and $q_{1,1} < q_{2,1}$.

Repeating the argument, for each $j = 2, 3, \dots$, we can find a subsequence $\{p^{(q_{j,i})}(x, 1/(j+1))\}_{i=1}^\infty$ of $\{p^{(q_{j-1,i})}(x, 1/(j+1))\}_{i=1}^\infty$ with $q_{j,1} > q_{j-1,1}$ and a function $p_j \in L^\infty_{\text{loc}}(R^n)$ such that $p^{(q_{j,i})}(x, 1/(j+1)) \to p_j(x)$ weakly in $(L^\infty(K))^*$ for every compact set $K \subset R^n$ as $i \to \infty$ and $p^{(q_{j,i})}(x, 1/(j+1)) \to p_j(x)$ a.e. $x \in R^n$ as $i \to \infty$.

Let $q'_i = q_{i,i}$. Then for each $j = 1, 2, \dots$, $\{p^{(q'_i)}(\cdot, 1/(j+1))\}_{i=1}^\infty$ is a subsequence of $\{p^{(q_{j,i})}(x, 1/(j+1))\}_{i=1}^\infty$. Hence $p^{(q'_i)}(x, 1/(j+1)) \to p_j(x)$ weakly in $(L^\infty(K))^*$ for every compact set $K \subset R^n$ as $i \to \infty$ and $p^{(q'_i)}(x, 1/(j+1)) \to p_j(x)$ a.e. $x \in R^n$ as $i \to \infty$. Thus $\{p_j\}_{j=1}^\infty$ is also uniformly bounded on every compact subset of R^n. So there exists a subsequence $\{p_{j_k}\}_{k=1}^\infty$ of $\{p_j\}_{j=1}^\infty$ and a function $\tilde{g} \in L^\infty_{\text{loc}}(R^n)$ such that $p_{j_k} \to \tilde{g}$ weakly in $(L^\infty(K))^*$ for any compact subset $K \subset R^n$. Letting $\varepsilon_k = 1/(j_k + 1)$, the lemma follows.

2

In this section we will first establish some technical lemmas and prove the main theorem (Theorem 2.10) under the assumption that $f \in C_0^1(R^n)$ (Theorem 2.9) The main theorem will then follow by an approximation argument.

Theorem 2.1. *Suppose $f \in C_0(R^n)$. For any sequence $\{u^{(q_i)}\}_{i=1}^{\infty}$, $q_i \to \infty$ as $i \to \infty$, of $\{u^{(q)}\}_{q>m+1}$, there exists a subsequence $\{u^{(q_i')}\}_{i=1}^{\infty}$ of $\{u^{(q_i)}\}_{i=1}^{\infty}$ and a $u^{(\infty)} \in C(R^n \times (0, 1))$, $0 \le u^{(\infty)} \le 1$, such that $u^{(q_i')} \to u^{(\infty)}$ uniformly on compact subsets $R \times (0, 1)$ as $i \to \infty$. Moreover $u^{(\infty)}$ satisfies* (0.2) *with initial trace $g \in L^1(R^n)$, $0 \le g \le 1$, satisfying* (0.3) *for some function $\widetilde{g} \in L^{\infty}_{\text{loc}}(R^n)$, $\widetilde{g} \ge 0$.*

Proof. The proof is a modification of the proof of Theorem 4 of [H1]. We first observe that $u^{(q)}$ is uniformly bounded by $\|f\|_{L^\infty}$ by Theorem 1.3 and there exists $R' > 0$ such that

(2.1)
$$0 \le u^{(q)}(x, t) \le \left(\frac{|x_1| + R' + 1}{t}\right)^{1/q-1} \quad \forall x = (x_1, x') \in R^n, \quad 0 < t < 1, q > m + 1,$$

by Lemma 1.6. If $\gamma(s) = s^{q/m}/q$, then $\gamma(u^{(q)m}) = \frac{u^{(q)q}}{q}$ and

$$\begin{aligned} \gamma'(u^{(q)m}(x, t)) &= \frac{1}{m}(u^{(q)}(x, t))^{q-m} \\ &\le \frac{1}{m}\left(\frac{|x_1| + R' + 1}{t}\right)^{q-m/q-1} \\ &\le \frac{1}{m}\left(\frac{|x_1| + R' + 1}{t}\right) \quad \forall x = (x_1, x') \in R^n, 0 < t < 1, q > m + 1, \end{aligned}$$

by (2.1). Hence both $u^{(q)m}$ and $\gamma'(u^{(q)m})$ are uniformly bounded on compact subsets of $R^n \times (0, 1)$ for $q > m+1$. By the result of P. Sacks [S1], $\{u^{(q)}\}_{q>m+1}^{\infty}$ is uniformly Hölder continuous on every compact subset of $R^n \times (0, 1)$. Hence $\{u^{(q_i)}\}_{i=1}^{\infty}$ has a convergent subsequence $\{u^{(q_i')}\}_{i=1}^{\infty}$ such that $\{u^{(q_i')}\}_{i=1}^{\infty}$ converges uniformly on every compact subset of $R^n \times (0, 1)$. Without loss of generality we may assume that $\{u^{(q_i)}\}_{i=1}^{\infty}$ converges uniformly on every compact subset of $R^n \times (0, 1)$. Let $u^{(\infty)} = \lim_{i\to\infty} u^{(q_i)}$. Then $u^{(\infty)} \in C(R^n \times (0, 1))$. Putting $q = q_i$ and letting $i \to \infty$ in (2.1), we get $0 \le u^{(\infty)} \le 1$. Putting $h(u) = u^{q_i}/q_i$, $u = u^{(q_i)}$ in (0.5) and letting $i \to \infty$ we see that, by Lemma 1.6, $u^{(\infty)}$ satisfies

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} \left[u^m \Delta\eta + u\frac{\partial \eta}{\partial t}\right] dx dt = \int_{\tau_1}^{\tau_2} \int_{\partial\Omega} u^m \frac{\partial \eta}{\partial N} d\sigma ds + \int_{\Omega} u\eta dx \bigg|_{\tau_1}^{\tau_2} \tag{2.2}$$

for all bounded open sets $\Omega \subset R^n$ with $\partial\Omega \in C^2$, $0 < \tau_1 \le \tau_2 < 1$, $\eta \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\eta \equiv 0$ on $\partial\Omega \times [\tau_1, \tau_2]$. Hence $u^{(\infty)}$ is a solution of the equation $u_t = \Delta u^m$ in $R^n \times (0, 1)$. Since $\|u^{(\infty)}\|_{L^\infty} \le \|f\|_{L^\infty}$, $u^{(\infty)}$ has an initial trace $d\mu$ by [DK] and $d\mu$ is absolutely continuous with respect to the Lebesgue measure. Hence $d\mu = g(x)dx$ for some function $g \ge 0$. Since $0 \le u^{(\infty)} \le 1$ and

$$\lim_{t\to 0} u^{(\infty)}(x, t) = g(x) \quad \text{a.e. } x \in R^n \tag{2.3}$$

by the result of [DFK], $0 \le g \le 1$. Since

$$\int_{R^n} u^{(q_i)}(x, t) dx = \int_{R^n} f(x) dx, \quad \forall 0 < t \le 1, i = 1, 2, \ldots.$$

Letting $i \to \infty$, we get by Fatou's lemma,

$$\int_{R^n} u^{(\infty)}(x, t)dx \le \int_{R^n} f(x)dx, \quad \forall 0 < t \le 1.$$

Letting $t \to 0$, we get by Fatou's lemma and (2.3),

$$\int_{R^n} g(x)dx \le \int_{R^n} f(x)dx.$$

Hence $g \in L^1(R^n)$. Let $p^{(q)}$ be as in Lemma 1.8 and Ω be a bounded open subset of R^n with $\partial\Omega \in C^2$. Then by Lemma 1.8 there exists a constant $C_1 > 0$ such that $\|p^{(q)}\|_{L^\infty(\overline{\Omega}\times[0,1))} \le C_1$ for all $q > m+1$ and there exists a subsequence $\{p^{(q_i')}\}_{i=1}^\infty$ of $\{p^{(q_i)}\}_{i=1}^\infty$, a sequence of functions $\{p_j\}_{j=1}^\infty \subset L^\infty_{\text{loc}}(R^n)$, $\widetilde{g} \in L^\infty_{\text{loc}}(R^n)$, $p_j, \widetilde{g} \ge 0$, and a sequence $\{\varepsilon_j\}_{j=1}^\infty \subset R$, $\varepsilon_j \to 0$ as $j \to \infty$, such that (1.18) holds. Hence for any $0 < \tau_2 < 1$, $\eta \in C_0^\infty(R^n)$,

$$\begin{aligned}\left|\int_0^{\tau_2}\int_\Omega \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau - \int_\Omega \widetilde{g}\eta_{x_1}dx\right| &\le \left|\int_\Omega\int_{\varepsilon_j}^{\tau_2} \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau\right| \\ &\quad + \left|\int_\Omega\left(\int_0^{\varepsilon_j} \frac{u^{(q_i')q_i'}(x, \tau)}{q_i'}d\tau\right)\eta_{x_1}(x)dx - \int_{R^n}\widetilde{g}\eta_{x_1}dx\right| \\ &\le \|\eta_{x_1}\|_{L^\infty(R^n)}\int_\Omega\int_{\varepsilon_j}^{\tau} \frac{u^{(q_i')q_i'}}{q_i'}dxd\tau \\ &\quad + \left|\int_\Omega p^{(q_i')}(x, \varepsilon_j)\eta_{x_1}(x)dx - \int_\Omega p_j(x)\eta_{x_1}(x)dx\right| \\ &\quad + \left|\int_\Omega p_j(x)\eta_{x_1}(x)dx - \int_\Omega \widetilde{g}(x)\eta_{x_1}(x)dx\right|.\end{aligned}$$

Letting first $i \to \infty$ and then $j \to \infty$, we get by Lemma 1.7 and Lemma 1.8,

$$\limsup_{i\to\infty}\left|\int_0^{\tau_2}\int_\Omega \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau - \int_\Omega \widetilde{g}\eta_{x_1}dx\right| = 0$$

$$\Rightarrow \quad \lim_{i\to\infty}\int_0^{\tau_2}\int_\Omega \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau = \int_\Omega \widetilde{g}\eta_{x_1}dx. \tag{2.4}$$

Putting $h(u) = u^{q_i'}/q_i'$, $u = u^{(q_i')}$, in (0.5) and letting $\tau_1 \to 0$, we have

$$\begin{aligned}\int_0^{\tau_2}\int_{R^n} u^{(q_i')m}\Delta\eta dxd\tau + \int_0^{\tau_2}\int_{R^n}\frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau \\ = \int_{R^n} u^{(q_i')}(x, \tau_2)\eta(x)dx - \int_{R^n} f\eta dx\end{aligned}$$

for all $\eta \in C_0^\infty(R^n)$, $0 < \tau_2 < 1$. Letting $i \to \infty$, we get by (2.4) and Lebesgue dominated convergence theorem,

$$\int_0^{\tau_2}\int_{R^n} u^{(\infty)m}\Delta\eta dxd\tau + \int_{R^n}\widetilde{g}\eta_{x_1}dxd\tau = \int_{R^n} u^{(\infty)}(x, \tau_2)\eta(x)dx - \int_{R^n} f\eta dx$$

for all $\eta \in C_0^\infty(R^n)$. Letting $\tau_2 \to 0$,

$$\begin{aligned}&\int \widetilde{g}\eta_{x_1}dx = \int g\eta dx - \int f\eta dx \quad \forall \eta \in C_0^\infty(R^n) \\ &\Rightarrow g + \widetilde{g}_{x_1} = f \quad \text{in } \mathscr{D}'(R^n).\end{aligned}$$

This completes the proof of Theorem 2.1.

We will now let

$$S(g) = \left\{x_0 \in R^n : \lim_{h\to 0} \frac{1}{|B_h(0)|} \int_{B_h(x_0)} |g(x) - g(x_0)|dx = 0\right\},$$

$$G(u^{(\infty)}, g) = \left\{x \in R^n : \lim_{t\to 0} u^{(\infty)}(x, t) = g(x)\right\}.$$

Lemma 2.2. *Let f, $u^{(\infty)}$, $u^{(q_i')}$, g be as in Theorem 2.1 and let $S^* = S(g) \cap G(u^{(\infty)}, g) \cap \{g < 1\}$. If $x_0 \in S^*$ is such that $g(x_0) \le \theta < 1$, then for any $\theta_1 \in (\theta, 1)$ and $\delta > 0$, there exists $q_0 > m+1$, $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$, such that*

$$\inf_{|x-x_0|\le\delta} u^{(q_i')}(x, t) \le \theta_1 \quad \forall 0 < t \le \varepsilon_0, \; q_i' \ge q_0.$$

Proof. The proof is similar to the proof of Theorem 3.3 of [CF]. Suppose the lemma is not true. Then there exists $\theta_1 \in (\theta, 1)$, $\delta > 0$, and $\{\varepsilon_i\}_{i=1}^{\infty}$, $0 < \varepsilon_i < 1/2$, $i = 1, 2, \dots$, $\varepsilon_i \to 0$ as $i \to \infty$ and a subsequence $\{u^{(q_i'')}\}_{i=1}^{\infty}$ of $\{u^{(q_i')}\}_{i=1}^{\infty}$ such that

$$\inf_{|x-x_0|\le\delta} u^{(q_i'')}(x, \varepsilon_i) > \theta_1.$$

Let $\widetilde{u}^{(q_i'')}$ be the solution of (1.1) in $R^n \times (0, 1)$ with initial value $\widetilde{u}^{(q_i'')}(x, 0) = \theta_1\chi_{B_\delta(x_0)}$ where $\chi_{B_\delta(x_0)}$ is the characteristic function of the set $B_\delta(x_0)$. By Theorem 1.1,

$$\begin{aligned}
&\tilde{u}^{(q_i'')}(x, t) \le u^{(q_i'')}(x, t+\varepsilon_i) \quad \forall x \in R^n, 0 < t \le 1/2\\
&\quad \Rightarrow \iint \widetilde{u}^{(q_i'')}(x, t)\eta(x, t)dxdt \le \iint u^{(q_i'')}(x, t+\varepsilon_i)\eta(x, t)dxdt\\
&\quad = \iint u^{(q_i'')}(x, t)\eta(x, t-\varepsilon_i)dxdt \qquad (2.5)
\end{aligned}$$

for all $\eta \in C_0^\infty(R^n \times (0, 1/2))$ and ε_i sufficiently small. By Theorem 2.1, $\{\widetilde{u}^{(q_i'')}\}_{i=1}^{\infty}$ has a convergent subsequence converging uniformly on compact subsets of $R^n \times (0, 1)$. Without loss of generality, we may assume that $\{\widetilde{u}^{(q_i'')}\}_{i=1}^{\infty}$ converges uniformly on compact subsets of $R^n \times (0, 1)$. Let $\widetilde{u}^{(\infty)} = \lim_{i\to\infty} \widetilde{u}^{(q_i'')}$. Since $0 \le u^{(q_i'')} \le \theta_1 < 1$, letting $i \to \infty$ in (2.5), we get by Lebesgue dominated convergence theorem

$$\begin{aligned}
&\iint \tilde{u}^{(\infty)}(x, t)\eta(x, t)dxdt \le \iint u^{(\infty)}(x, t)\eta(x, t)dxdt\\
&\hspace{18em} \forall \eta \in C_0^\infty(R^n \times (0, 1/2))\\
&\quad \Rightarrow \widetilde{u}^{(\infty)}(x, t) \le u^{(\infty)}(x, t) \quad \forall x \in R^n, 0 < t < 1/2\\
&\hspace{14em} \text{since } u^{(\infty)}, \widetilde{u}^{(\infty)} \in C(R^n \times (0, 1/2))\\
&\quad \Rightarrow \int_{R^n} \widetilde{u}^{(\infty)}(x, t)\eta(x)dx \le \int_{R^n} u^{(\infty)}(x, t)\eta(x)dx \qquad \forall \eta \in C_0^\infty(R^n)\\
&\quad \Rightarrow \int_{R^n} \theta_1\chi_{B_\delta(x_0)}(x)dx \le \int_{R^n} g(x)\eta(x)dx \qquad \text{as } t \to 0 \quad \forall \eta \in C_0^\infty(R^n)\\
&\quad \Rightarrow \theta < \theta_1 \le g(x_0)
\end{aligned}$$

since $x_0 \in S(g)$. Thus contradiction arise and the lemma follows.

Lemma 2.3. *Suppose* $f \in C_0^1(R^n)$. *Let* $u^{(\infty)}$, $u^{(q_i')}$, g *be as in Theorem* 2.1 *and let* S^* *be as in Lemma* 2.2. *If* $x_0 \in S^*$ *is such that* $g(x_0) \le \theta < 1$, *then for any* $\theta_1 \in (\theta, 1)$, *there exists* $q_0 > m+1$, $\varepsilon_0 \in (0, 1/2)$ *depending only on* θ, θ_1 *and* $\|f_{x_k}\|_{L^\infty(R^n)}$, $k = 1, 2, \dots, n$, *such that*

$$u^{(q_i')}(x, t) \le \theta_1 \quad \forall x \in B_\delta(x_0), 0 < t \le \varepsilon_0, q_i' \ge q_0,$$

where $\delta = (\theta_1 - \theta)/4(\sqrt{n}\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)} + 1)$.

Proof. The proof is similar to the proof of Theorem 2.4 of [H2]. Let

$$\delta = (\theta_1 - \theta)/4(\sqrt{n}\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)} + 1).$$

Then by Lemma 2.2, there exists $q_0 > m+1$, $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$, such that

$$\inf_{|x-x_0|\le\delta} u^{(q_i')}(x, t) \le \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \le \varepsilon_0, q_i' \ge q_0.$$

Hence for each $q_i' \ge q_0$ and $0 < t \le \varepsilon_0$, there exists an $x_t \in \overline{B_\delta(x_0)}$ such that

$$u^{(q_i')}(x_t, t) \le \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \le \varepsilon_0.$$

For any $0 < \varepsilon < 1$, let $f_\varepsilon = f + \varepsilon$ and let $u_\varepsilon^{(q)}$ be the solution of (1.1) in $R^n \times (0, 1)$ with $u_\varepsilon^{(q)}(x, 0) = f_\varepsilon(x)$ given by Lemma 1.5. Then by Lemma 1.5,

$$\begin{aligned}
&|u_\varepsilon^{(q_i')}(x, t) - u_\varepsilon^{(q_i')}(x_t, t)| \\
&= \left|\int_0^1 \frac{d}{ds} u_\varepsilon^{(q_i')}(sx + (1-s)x_t, t)\,ds\right| \\
&\le \int_0^1 |\nabla u_\varepsilon^{(q_i)}(sx + (1-s)x_t, t)\cdot(x - x_t)|\,ds \\
&\le \sqrt{n}\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)}|x - x_t| \\
&\le 2\delta\sqrt{n}\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)} \le (\theta_1 - \theta)/2 \\
&\Rightarrow u_\varepsilon^{(q_i')}(x, t) \le u_\varepsilon^{(q_i')}(x_t, t) + (\theta_1 - \theta)/2 \le (\theta_1 + \theta)/2 + (\theta_1 - \theta)/2 \le \theta_1
\end{aligned}$$

for all $x \in B_\delta(x_0)$, $0 < t \le \varepsilon_0$, $q_i' \ge q_0$. Since $u_\varepsilon^{(q_i')} \to u^{(q_i')}$ uniformly on compact subsets of $R^n \times (0, 1)$ as $\varepsilon \to 0$ by Lemma 1.5, letting $\varepsilon \to 0$ we get

$$u^{(q_i')}(x, t) \le \theta_1 \quad \forall x \in B_\delta(x_0), 0 < t \le \varepsilon_0, q_i' \ge q_0.$$

Lemma 2.4. *Suppose* $f \in C_0^1(R^n)$. *Let* g, $\tilde{g}$ *be as in Theorem* 2.1 *and let* S^* *be as in Lemma* 2.2. *Then*

$$\begin{cases} g(x) = f(x), \\ \tilde{g}(x) = 0 \end{cases}$$

for all $x \in S^* \cap S(\tilde{g})$.

Proof. Let $u^{(\infty)}$, $u^{(q_i)}$ be as in Theorem 2.1. By Theorem 2.1 we may assume without loss of generality that $u^{(q_i)}$ converges uniformly to $u^{(\infty)}$ on compact subsets of $R^n \times (0, 1)$ as $i \to \infty$. We also let $p^{(q_i)}$, $p^{(q_i')}$, p_j, ε_j be as in Lemma 1.8. Suppose $x_0 \in S^* \cap S(\tilde{g})$. Then there exists θ, $\theta_1 > 0$ such that

$g(x_0)\le\theta<\theta_1<1$. By Lemma 2.3, there exists $q_0>m+1$, $\delta>0$, $\varepsilon_0>0$, $0<\varepsilon_0<1/2$ such that

$$u^{(q_i')}(x,t)\le\theta_1\quad\forall x\in B_\delta(x_0),\,0<t\le\varepsilon_0,\,q_i'\ge q_0.$$

Hence

$$\begin{aligned}&\left|\int_{R^n}u^{(q_i')}(x,t)\eta(x)dx-\int_{R^n}f(x)\eta(x)dx\right|\\&=\left|\int_0^t\int_{R^n}[u^{(q_i')m}\Delta\eta+\frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}]dxd\tau\right|\\&\le\theta_1^m\|\Delta\eta\|_{L^1(R^n)}t+\frac{\theta_1^{q_i'}}{q_i'}\|\eta_{x_1}\|_{L^1(R^n)}t\quad\forall q_i'\ge q_0,\,0<t\le\varepsilon_0,\,\eta\in C_0^\infty(B_\delta(x_0)).\end{aligned}$$

Letting $i\to\infty$,

$$\begin{aligned}&\left|\int_{R^n}u^{(\infty)}(x,t)\eta(x)dx-\int_{R^n}f(x)\eta(x)dx\right|\\&\le\theta_1^m\|\Delta\eta\|_{L^1(R^n)}t+\frac{\theta_1^{q_i'}}{q_i'}\|\eta_{x_1}\|_{L^1(R^n)}t\quad\forall q_i'\ge q_0,\,0<t\le\varepsilon_0,\,\eta\in C_0^\infty(B_\delta(x_0)).\end{aligned}$$

Letting $t\to0$,

$$\int_{R^n}g\eta dx=\int_{R^n}f\eta dx\quad\forall\eta\in C_0^\infty(B_\delta(x_0))\Rightarrow g(x_0)=f(x_0)$$

since $x_0\in S(g)$. Similarly

$$\begin{aligned}&\int_{B_\delta(x_0)}p^{(q_i')}(x,\varepsilon_j)dx\\&\quad=\int_{B_\delta(x_0)}\int_0^{\varepsilon_j}\frac{u^{(q_i')q_i'}}{q_i'}d\tau dx\le\frac{\theta_1^{q_i'}}{q_i'}|B_\delta(x_0)|\varepsilon_j\to0\text{ as }i\to0\quad\forall j=1,2,\dots\\&\quad\Rightarrow\int_{B_\delta(x_0)}p_j(x)dx=0\qquad\text{by Fatou's lemma since }p_j\ge0\\&\quad\Rightarrow\int_{B_\delta(x_0)}\tilde g(x)dx=0\qquad\text{by Fatou's lemma since }\tilde g\ge0\\&\quad\Rightarrow\tilde g\equiv0\text{ on }B_\delta(x_0)\\&\quad\Rightarrow\tilde g(x_0)=0\text{ since }x_0\in S(\tilde g).\end{aligned}$$

Lemma 2.5. *Suppose $f\in C_0^1(R^n)$ and let g, $\tilde g$ be as in Theorem 2.1. Then there exists $r'>0$ such that*

$$\begin{cases}g(x)=f(x),\\\tilde g(x)=0\end{cases}\tag{2.6}$$

a.e. $x\in R^n\setminus B_{r'}(0)$

Proof. Let $u^{(\infty)}$, $u^{(q_i')}$ be as in Theorem 2.1, S^* be as in Lemma 2.2 and let $S_1=S(g)\cap S(\tilde g)\cap G(u^{(\infty)},g)$. $S_2=S(g)\cap G(u^{(\infty)},g)$. For any $0<\theta<1$, $r>0$, let $A_{\theta,r}=\{x\in R^n\setminus B_r(0):g(x)\ge\theta\}$. We now fix θ, $\theta_1\in(0,1)$ such that $\theta<\theta_1$. Choose a constant $\theta'>0$ such that $\theta<\theta'<\theta_1$ and let

$$\delta=\min((\theta'-\theta)/4(\sqrt n\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)}+1),1).$$

Since $g \in L^1(R^n)$,

$$\int_{|x|\ge r} g\,dx \to 0 \text{ as } r \to 0.$$

Thus there exists $r_0 > 0$ such that

$$\begin{aligned}\int_{|x|\ge r_0} g\,dx &\le \frac{1}{2}\theta|B_\delta(0)| \\ \Rightarrow \theta|A_{\theta,r_0}| &\le \frac{1}{2}\theta|B_\delta(0)| \\ \Rightarrow |A_{\theta,r_0}| &\le \frac{1}{2}|B_\delta(0)|.\end{aligned} \tag{2.7}$$

Let $r' = r_0 + 1$. Since $|R^n \setminus S_1| = 0$ by the result of [DFK] and Chapter 1 of [S], (2.6) holds for a.e. $x \in A^c_{\theta,r'}$ by Lemma 2.4. Hence in order to prove the lemma, it suffices to show that (2.6) holds for a.e. $x \in A_{\theta,r'} \cap S_1$. Let $y_0 \in A_{\theta,r'} \cap S_1$. If $|B_\delta(y_0) \cap A^c_{\theta,r'}| = 0$, then

$$g(z) \ge \theta \text{ a.e. } z \in B_\delta(y_0) \Rightarrow |A_{\theta,r_0}| \ge |B_\delta(y_0)|$$

since $B_\delta(y_0) \subset R^n \setminus B_{r_0}(0)$. This contradicts (2.7). Thus $|B_\delta(y_0) \cap A^c_{\theta,r'}| \ne 0$. Since $|(B_\delta(y_0) \cap A^c_{\theta,r'}) \setminus (B_\delta(y_0) \cap A^c_{\theta,r'} \cap S_2)| = 0$, $B_\delta(y_0) \cap A^c_{\theta,r'} \cap S_2 \ne \varnothing$ and there exists $x_0 \in B_\delta(y_0) \cap A^c_{\theta,r'} \cap S_2 \subset S^*$. By Lemma 2.3, there exists $q_0 > m+1$ and $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$, such that

$$u^{(q_i')}(x,t) \le \theta' \quad \forall x \in B_\delta(x_0),\ 0 < t \le \varepsilon_0,\ q_i' \ge q_0.$$

Letting $i \to \infty$,

$$\begin{aligned}&u^{(\infty)}(x,t) \le \theta' \quad \forall x \in B_\delta(x_0),\ 0 < t \le \varepsilon_0 \\ &\Rightarrow \int_{R^n} u^{(\infty)}(x,t)\eta(x)\,dx \le \theta' \int_{R^n} \eta\,dx \quad \forall \eta \in C_0(B_\delta(x_0)) \\ &\Rightarrow \int_{R^n} g\eta\,dx \le \theta' \int_{R^n} \eta\,dx \quad \forall \eta \in C_0(B_\delta(x_0)) \qquad \text{as } t \to 0 \\ &\Rightarrow g(y_0) \le \theta' < 1\end{aligned}$$

since $y_0 \in S(g) \cap B_\delta(x_0)$. Hence $y_0 \in S^* \cap S(\tilde{g})$. Thus (2.6) holds for $x = y_0$ by Lemma 2.4 and the lemma follows.

Corollary 2.6. *Suppose $f \in C^1_0(R^n)$ and $\tilde{g}$ is as in Theorem 2.1. Then $\tilde{g} \in L^1(R^n)$.*

Proof. The lemma follows directly from Lemma 2.5 and the fact that $\tilde{g} \in L^\infty_{\text{loc}}(R^n)$.

Lemma 2.7. *For any $0 \le f_1, f_2, g_1, g_2, \tilde{g}_1, \tilde{g}_2 \in L^1(R^n)$, $0 \le g_1, g_2 \le 1$, $\tilde{g}_1, \tilde{g}_2 \ge 0$, if*

$$g_i + (\tilde{g}_i)_{x_1} = f_i \qquad \text{in } \mathscr{D}'(R^n) \tag{2.8}$$

and

$$g_i(x) = f_i(x),\ \tilde{g}_i(x) = 0 \qquad \text{whenever } g_i(x) < 1 \text{ a.e. } x \in R^n \tag{2.9}$$

for $i = 1, 2$, then

$$\int_{|x_1|\le R'} \int_{R^{n-1}} |\tilde{g}_1 - \tilde{g}_2|(x_1, x')\,dx'\,dx_1 \le 2R'\|f_1 - f_2\|_{L^1(R^n)} \quad \forall R' > 0.$$

Proof. We will use a modification of an argument of [SX]. By (2.8),

$$
\begin{aligned}
&(g_1 - g_2) + (\widetilde{g}_1 - \widetilde{g}_2)_{x_1} = f_1 - f_2 \qquad \text{in } \mathscr{D}'(R^n)\\
&\Rightarrow \int_{R^n} [(g_1 - g_2)\eta - (\widetilde{g}_1 - \widetilde{g}_2)\eta_{x_1}]dx\\
&= \int_{R^n} (f_1 - f_2)\eta dx \quad \forall \eta \in C_0^\infty(R^n).
\end{aligned} \tag{2.10}
$$

Putting $\eta(x) = \rho_\varepsilon(\xi - x)$ in (2.10), we get

$$
(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})_{\xi_1}(\xi) = (f_{1,\varepsilon} - f_{2,\varepsilon})(\xi) - (g_{1,\varepsilon} - g_{2,\varepsilon})(\xi) \quad \forall \xi = (\xi_1, \dots, \xi_n) \in R^n. \tag{2.11}
$$

For any $k = 1, 2, \dots$, we let $p_k(\cdot) \in C_0^\infty(R)$, $0 \le p_k \le 1$, be such that $p_k(x) \equiv 1$ for $x \ge 1/k$, $p_k(x) \equiv 0$ for $x \le 1/2k$ and $\|p_{k,x}\|_{L^\infty} \le 5k$. Then for all $z_1, y_1 \in R$,

$$
\begin{aligned}
&\int_{R^{n-1}} (\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(z_1, x')p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(z_1, x')dx'\\
&\quad - \int_{R^{n-1}} (\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(y_1, x')p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(y_1, x')dx'\\
&\quad = \int_{R^{n-1}}\int_{y_1}^{z_1} \frac{\partial}{\partial x_1}\left[(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(x_1, x')p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(x_1, x')\right]dx_1dx'\\
&\quad = \int_{R^{n-1}}\int_{y_1}^{z_1} \left[\left(\frac{\partial}{\partial x_1}(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})\right)p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})\right.\\
&\qquad\qquad \left. + (\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})p_k'(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})\frac{\partial}{\partial x_1}(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})\right]dx_1dx'
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \int_{R^{n-1}} (\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(z_1, x')p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(z_1, x')dx'\\
&\quad + \int_{R^{n-1}}\int_{y_1}^{z_1} (g_{1,\varepsilon} - g_{2,\varepsilon})(x_1, x')p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(x_1, x')dx_1dx'\\
&\quad = \int_{R^{n-1}} (\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(y_1, x')p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(y_1, x')dx'\\
&\quad + \int_{R^{n-1}}\int_{y_1}^{z_1} (f_{1,\varepsilon} - f_{2,\varepsilon})(x_1, x')p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(x_1, x')dx_1dx'\\
&\quad + \int_{R^{n-1}}\int_{y_1}^{z_1} (\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})p_k'(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})\\
&\qquad\qquad \cdot[(f_{1,\varepsilon} - f_{2,\varepsilon}) - (g_{1,\varepsilon} - g_{2,\varepsilon})](x_1, x')dx_1dx'
\end{aligned} \tag{2.12}
$$

by (2.11). Since $\widetilde{g_1}, \widetilde{g_2} \in L^1(R^n)$,

$$
\int_{R^n} |\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon}|\cdot|p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})|dx \le \int_{R^n}(\widetilde{g}_1 + \widetilde{g}_2)dx < \infty.
$$

Hence there exists a sequence $\{y_1^j\}_{j=1}^\infty \subset R$, $y_1^j \to -\infty$ as $j\to\infty$ such that

$$
\int_{R^{n-1}} (\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(y_1^j, x')p_k(\widetilde{g}_{1,\varepsilon} - \widetilde{g}_{2,\varepsilon})(y_1^j, x')dx' \to 0 \qquad \text{as } j\to\infty.
$$

Putting $y_1 = y_1^j$ in (2.12) and letting $j \to \infty$, we get

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x') dx' \\
&\qquad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_{1,\varepsilon} - g_{2,\varepsilon})(x_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x') dx_1 dx' \\
&\quad = \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) \\
&\qquad\qquad \cdot [(f_{1,\varepsilon} - f_{2,\varepsilon}) - (g_{1,\varepsilon} - g_{2,\varepsilon})](x_1, x') dx_1 dx' \\
&\qquad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (f_{1,\varepsilon} - f_{2,\varepsilon})(x_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x') dx_1 dx'.
\end{aligned} \tag{2.13}$$

Since $\tilde{g}_1, \tilde{g}_2 \in L^1(R^n)$,

$$\begin{aligned}
&\int_R \left| \int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x') dx' \right. \\
&\qquad\qquad \left. - \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)(z_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(z_1, x') dx' \right| dz_1 \\
&\quad \le \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - (\tilde{g}_1 - \tilde{g}_2)| p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) dx \\
&\qquad + \int_{R^n} (\tilde{g}_1 - \tilde{g}_2) \cdot |p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - p_k(\tilde{g}_1 - \tilde{g}_2)| dx \\
&\quad \le \int_{R^n} |\tilde{g}_{1,\varepsilon} - \tilde{g}_1| dx + \int_{R^n} |\tilde{g}_{2,\varepsilon} - \tilde{g}_2| dx \\
&\qquad + \int_{R^n} (\tilde{g}_1 + \tilde{g}_2) \cdot |p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - p_k(\tilde{g}_1 - \tilde{g}_2)| dx \\
&\quad \to 0 \qquad \text{as } \varepsilon \to 0
\end{aligned}$$

by the Lebesgue dominated convergence theorem and Theorem 2 in Chapter 3 of [S]. Hence there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty} \subset R$, $\varepsilon_j \to 0$ as $j \to \infty$, such that

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_{1,\varepsilon_j} - \tilde{g}_{2,\varepsilon_j})(z_1, x') p_k(\tilde{g}_{1,\varepsilon_j} - \tilde{g}_{2,\varepsilon_j})(z_1, x') dx' \\
&\qquad \to \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)(z_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(z_1, x') dx'
\end{aligned}$$

a.e. $z_1 \in R$ as $j \to \infty$. On the other hand,

$$\begin{aligned}
&\left| \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) \right. \\
&\qquad \cdot [(g_{1,\varepsilon} - g_{2,\varepsilon}) - (f_{1,\varepsilon} - f_{2,\varepsilon})](x_1, x')dx_1dx' \\
&\quad \left. - \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_1 - \tilde{g}_2) p_k'(\tilde{g}_1 - \tilde{g}_2)[(g_1 - g_2) - (f_1 - f_2)](x_1, x')dx_1dx' \right| \\
&\le \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})||(g_{1,\varepsilon} - g_{2,\varepsilon}) - (g_1 - g_2)|dx \\
&\quad + \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})||(f_{1,\varepsilon} - f_{2,\varepsilon}) - (f_1 - f_2)|dx \\
&\quad + \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - (\tilde{g}_1 - \tilde{g}_2) p_k'(\tilde{g}_1 - \tilde{g}_2)||g_1 - g_2|dx \\
&\quad + \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - (\tilde{g}_1 - \tilde{g}_2) p_k'(\tilde{g}_1 - \tilde{g}_2)||f_1 - f_2|dx \\
&\le 5 \int_{R^n} (|g_{1,\varepsilon} - g_1| + |g_{2,\varepsilon} - g_2| + |f_{1,\varepsilon} - f_1| + |f_{2,\varepsilon} - f_2|)dx \\
&\quad + \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - (\tilde{g}_1 - \tilde{g}_2) p_k'(\tilde{g}_1 - \tilde{g}_2)| \\
&\qquad \cdot (g_1 + g_2 + f_1 + f_2)dx \\
&\to 0 \qquad \text{as } \varepsilon \to 0
\end{aligned}$$

by the Lebesgue dominated convergence theorem since the integrand of the last integral above is bounded by $5(g_1 + g_2 + f_1 + f_2) \in L^1(R^n)$ and tends to 0 as $k \to \infty$. Similarly

$$\begin{aligned}
&\int_{R^{n-1}} \int_{-\infty}^{z_1} (g_{1,\varepsilon} - g_{2,\varepsilon})(x_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x')dx_1dx' \\
&\quad \to \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(x_1, x')dx_1dx' \qquad \text{as } \varepsilon \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{R^{n-1}} \int_{-\infty}^{z_1} (f_{1,\varepsilon} - f_{2,\varepsilon})(x_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x')dx_1dx' \\
&\quad \to \int_{R^{n-1}} \int_{-\infty}^{z_1} (f_1 - f_2)(x_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(x_1, x')dx_1dx' \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

Putting $\varepsilon = \varepsilon_j$ in (2.13) and letting $j \to \infty$, we get

(2.14)

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)(z_1, x') p_k (\tilde{g}_1 - \tilde{g}_2)(z_1, x') dx' \\
&\qquad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x') p_k (\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \\
&\quad = \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_1 - \tilde{g}_2) p_k' (\tilde{g}_1 - \tilde{g}_2)[(f_1 - f_2) - (g_1 - g_2)](x_1, x') dx_1 dx' \\
&\qquad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (f_1 - f_2)(x_1, x') p_k (\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \\
&\quad \le I_1 + \int_{R^n} (f_1 - f_2)_+ dx \qquad \text{a.e. } z_1 \in R
\end{aligned}$$

Since $p_k'(s) = 0$ for $s \le 1/2k$ or $s \ge 1/k$, I_1 is bounded by

$$\begin{aligned}
&\int_{R^n} |(\tilde{g}_1 - \tilde{g}_2)(x)| \, \|p_k'\|_{L^\infty} \cdot (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x_1, x') dx \\
&\quad \le \int_{R^n} \frac{1}{k} \cdot 5k \cdot (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x) dx \\
&\quad \le 5 \int_{R^n} (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x) dx \\
&\quad \to 0 \qquad \text{as } k \to \infty
\end{aligned}$$

by the Lebesgue dominated convergence theorem since $g_1, g_2, f_1, f_2 \in L^1(R^n)$ and

$$(g_1 + g_2 + f_1 + f_2)(x) \chi_{A_k}(x) \to 0 \quad \text{as } k \to \infty \text{ a.e. } x \in R^n$$

where $A_k = \{x \in R^n : 1/2k \le (g_1 - g_2)(x) \le 1/k\}$. Hence by letting $k \to \infty$ in (2.14), we get

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)_+ (z_1, x')(z_1, x') dx' \\
&\qquad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x') \operatorname{sign}_+ (\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \\
&\quad \le \int_{R^n} (f_1 - f_2)_+ dx
\end{aligned}$$

a.e. $z_1 \in R$. Since $(g_1 - g_2)(x) \operatorname{sign}_+(\tilde{g}_1(x) - \tilde{g}_2(x)) \ge 0$ a.e. $x \in R^n$ by (2.9),

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)_+ (z_1, x')(z_1, x') dx' \le \int_{R^n} (f_1 - f_2)_+ dx \qquad \text{a.e. } z_1 \in R \\
&\qquad \Rightarrow \int_{|x_1| \le R'} \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)_+ (x_1, x') dx' dx_1 \le 2R' \int_{R^n} (f_1 - f_2)_+ dx \quad \forall R' > 0.
\end{aligned}$$

Similarly

$$\int_{|x_1| \le R'} \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)_- (x_1, x') dx' dx_1 \le 2R' \int_{R^n} (f_1 - f_2)_- dx \quad \forall R' > 0.$$

Thus

$$\int_{|x_1| \le R'} \int_{R^{n-1}} |\tilde{g}_1 - \tilde{g}_2|(x_1, x')(x_1, x') dx' dx_1 \le 2R' \int_{R^n} |f_1 - f_2| dx \quad \forall R' > 0.$$

Corollary 2.8. *Let $0 \le f \in L^1(R^n)$. Then there exists at most one function g, $g \in L^1(R^n)$, $0 \le g \le 1$, and one function $\widetilde{g} \in L^1(R^n)$, $\widetilde{g} \ge 0$ satisfying*

$$\text{(2.15)} \qquad \begin{cases} g + (\widetilde{g})_{x_1} = f & \text{in } \mathscr{D}'(R^n), \\ g(x) = f(x),\ \widetilde{g}(x) = 0 & \text{whenever } g(x) < 1 \text{ a.e. } x \in R^n. \end{cases}$$

As a consequence of Theorem 2.1, Lemmas 2.4, 2.5, Corollary 2.8 and the uniqueness theorem (Theorem 6.13) of [DK], we have

Theorem 2.9. *Suppose $f \in C_0^1(R^n)$. Then there exists a unique function $u^{(\infty)} \in C(R^n \times (0, 1))$, $0 \le u^{(\infty)} \le 1$, such that $u^{(q)}$ converges uniformly to $u^{(\infty)}$ on compact subsets of $R^n \times (0, 1)$ as $q \to \infty$. Moreover $u^{(\infty)}$ satisfies* (0.2) *with initial value $g \in L^1(R^n)$, $0 \le g \le 1$, $\|g\|_{L^1(R^n)} \le \|f\|_{L^1(R^n)}$, satisfying* (2.15) *and* (2.6) *for some function $\tilde{g} \in L^1(R^n)$, $\tilde{g} \ge 0$.*

We are now ready to state and prove the main theorem.

Theorem 2.10. *For any $m > 1$ fixed, there exists a unique function $u^{(\infty)} \in C(R^n \times (0, 1))$, $0 \le u^{(\infty)} \le 1$ such that $u^{(q)}$ converges weakly to $u^{(\infty)}$ in $(L^\infty(G))^*$ for any compact subset G of $R^n \times (0, 1)$ as $q \to \infty$. Moreover $u^{(\infty)}$ satisfies (0.2) with initial value $g \in L^1(R^n)$, $0 \le g \le 1$, satisfying* (2.15) *for some function $\widetilde{g} \in L^1_{\text{loc}}(R^n)$, $\widetilde{g} \ge 0$. The convergence is uniform on every compact subsets of $R^n \times (0, 1)$ if $f \in C_0(R^n)$.*

Proof. Since $f \in L^\infty(R^n) \cap L^1(R^n)$, we can choose a sequence $\{f_j\}_{j=1}^\infty \subset C_0^1(R^n)$ such that $\|f_j\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)} + 1$, $\|f_j\|_{L^1(R^n)} \le \|f\|_{L^1(R^n)} + 1$ for all $j = 1, 2, \dots$ and $\|f_j - f\|_{L^1(R^n)} \to 0$ as $j \to \infty$.

For all $j = 1, 2, \dots$, let $u_j^{(q)}$ be the solution of (1.1) in $R^n \times (0, 1)$ with initial value $u_j^{(q)}(x, 0) = f_j(x)$. By Theorem 2.9, for each $j = 1, 2, \dots$, there exists an unique function $u_j^{(\infty)}$ such that $u_j^{(q)}$ converges uniformly on compact subsets of $R^n \times (0, 1)$ to $u_j^{(\infty)}$ as $q \to \infty$. Moreover $u_j^{(\infty)}$ satisfies (0.2) with initial value $g_j \in L^1(R^n)$, $0 \le g_j \le 1$, satisfying

$$\text{(2.16)} \qquad \begin{cases} \displaystyle\int_{R^n} g_j \le \int_{R^n} f_j \le \int_{R^n} f\,dx + 1, \\ g_j + (\widetilde{g}_j)_{x_1} = f \quad \text{in } \mathscr{D}'(R^n) \text{ for some } \widetilde{g}_j \in L^1(R^n),\ \widetilde{g}_j \ge 0, \\ g_j(x) = f(x),\ \widetilde{g}_j(x) = 0 \quad \text{whenever } g_j(x) < 1 \text{ a.e. } x \in R^n \end{cases}$$

for all $j = 1, 2, \dots$. Since $\|u^{(q)}\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)}$, any sequence $\{u^{(q_i)}\}_{i=1}^\infty$, $q_i \to \infty$ as $i \to \infty$, of $\{u^{(q)}\}_{q>1}$ has a subsequence $\{u^{(q_i')}\}_{i=1}^\infty$ such that $\{u^{(q_i')}\}_{i=1}^\infty$ converges weakly in $(L^\infty(G))^*$ for any compact subset G of $R^n \times (0, 1)$ as $i \to \infty$. Let $u^{(\infty)} = \lim_{i\to\infty} u^{(q_i')}$. Without loss of generality we may assume that $u^{(q_i')}(x, t) \to u^\infty(x, t)$ a.e. $(x, t) \in R^n \times (0, 1)$ as $i \to \infty$. By Theorem 1.4,

$$\int_{R^n} |u_j^{(q_i')} - u^{(q_i')}|(x, t)dx \le \int_{R^n} |f_j - f|(x)dx \quad \forall i, j = 1, 2, \dots$$
$$\Rightarrow \int_{\tau_1}^{\tau_2} \int_{R^n} |u_j^{(q_i')} - u^{(q_i')}|(x, t)dxdt$$
$$\le (\tau_2 - \tau_1) \int_{R^n} |f_j - f|(x)dx \quad \forall 0 < \tau_1 \le \tau_2 < 1.$$

Letting $i \to \infty$, we get by Fatou's lemma,

$$\int_{\tau_1}^{\tau_2}\int_{R^n}|u_j^{(\infty)}-u^{(\infty)}|(x,t)dxdt \le (\tau_2-\tau_1)\int_{R^n}|f_j-f|(x)dx \to 0 \text{ as } j\to\infty$$

for all $0<\tau_1\le\tau_2<1$.

Hence $u^{(\infty)}$ is the limit of the functions $\{u_j^{(\infty)}\}_{j=1}^\infty$ in $L^1_{\text{loc}}(R^n\times(0,1))$ as $j\to\infty$. Thus $u^{(\infty)}$ is unique and $u^{(q)}$ converges weakly to $u^{(\infty)}$ in $(L^\infty(G))^*$ for any compact subset G of $R^n\times(0,1)$ as $q\to\infty$. This together with Theorem 2.1 implies that $u^{(q)}$ converges uniformly to $u^{(\infty)}$ on every compact subsets of $R^n\times(0,1)$ as $q\to\infty$ if $f\in C_0(R^n)$.

Moreover $\{u_j^{(\infty)}\}_{j=1}^\infty$ has a subsequence converging a.e. $(x,t)\in R^n\times(0,1)$ to $u^{(\infty)}$. Without loss of generality we may assume that $u_j^{(\infty)}(x,t)\to u^{(\infty)}(x,t)$ a.e. $(x,t)\in R^n\times(0,1)$ as $j\to\infty$.

On the other hand since $u_j^{(\infty)}$ satisfies (0.2) and

$$\begin{aligned}&|u_j^{(q_i')}(x,t)|\le\|f_j\|_{L^\infty(R^n)}\le\|f\|_{L^\infty(R^n)}+1 \quad \forall(x,t)\in R^n\times(0,1),\\ &\qquad\qquad i,j=1,2,\dots,\\ &\Rightarrow \|u_j^{(\infty)}\|_{L^\infty(R^n\times(0,1))}\le\|f\|_{L^\infty(R^n)}+1 \quad \forall j=1,2,\dots.\end{aligned}\tag{2.17}$$

as $q\to\infty$ by Theorem 1.3, by the result of [S1] $\{u_j^{(\infty)}\}_{j=1}^\infty$ has a subsequence $\{u_{j_k}^{(\infty)}\}_{k=1}^\infty$ converging uniformly on compact subsets of $R^n\times(0,1)$. Hence we may assume without loss of generality that $\{u_j^{(\infty)}\}_{j=1}^\infty$ converges uniformly on compact subsets of $R^n\times(0,1)$ to $u^{(\infty)}$. Thus $u^{(\infty)}\in C(R^n\times(0,1))$.

Putting $h(u)=0$, $u=u_j^{(\infty)}$ in (0.5) and letting $j\to\infty$, we see that $u^{(\infty)}$ satisfies (0.2). By (2.17) and the result of [DK], $u^{(\infty)}$ has an initial trace $d\mu$ and $d\mu$ is absolutely continuous with respect to the Lebesgue measure. Hence $d\mu=g(x)dx$ for some $g\ge0$, $g\in L^1(R^n)$. By (2.16) and Lemma 2.7,

$$\int_{|x_1|\le R'}\int_{R^{n-1}}|\tilde g_j-\tilde g_{j'}|(x_1,x')dx'dx_1\le 2R'\|f_j-f_{j'}\|_{L^1(R^n)}\to0 \qquad \text{as } j,j'\to\infty \quad \forall R'>0.$$

Hence $\{\tilde g_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^1_{\text{loc}}(R^n)$ and there exists $\tilde g\in L^1_{\text{loc}}(R^n)$ such that $\tilde g_j\to\tilde g$ in $L^1_{\text{loc}}(R^n)$ as $j\to\infty$. Without loss of generality we may assume that $\tilde g_j(x)\to\tilde g(x)$ a.e. $x\in R^n$. By the proof of Theorem 2.1, $u_j^{(\infty)}$ satisfies, for all $\eta\in C_0^\infty(R^n)$, $0<\tau_2<1$,

$$\begin{aligned}&\int_0^{\tau_2}\int_{R^n}u_j^{(\infty)m}\Delta\eta\,dxd\tau+\int_{R^n}\tilde g_j\eta_{x_1}dx=\int_{R^n}u_j^{(\infty)}(x,t)\eta(x)dx-\int_{R^n}f_j\eta\,dx\\ &\Rightarrow\int_0^{\tau_2}\int_{R^n}u^{(\infty)m}\Delta\eta\,dxd\tau+\int_{R^n}\tilde g\eta_{x_1}dx\\ &=\int_{R^n}u^{(\infty)}(x,t)\eta(x)dx-\int_{R^n}f\eta\,dx \quad \text{as } j\to\infty\\ &\Rightarrow\int_{R^n}\tilde g\eta_{x_1}dx=\int_{R^n}g(x)\eta(x)dx-\int_{R^n}f\eta\,dx \quad \text{as } \tau_2\to0\\ &\Rightarrow g+\tilde g_{x_1}=f \quad \text{in } \mathscr{D}'(R^n).\end{aligned}$$

Thus

$$\begin{aligned}\left|\int (g - g_j)\eta dx\right| &= \left|\int (\tilde{g} - \tilde{g}_j)\eta_{x_1} dx + \int (f - f_j)\eta dx\right| \\ &\le \|\eta_{x_1}\|_{L^\infty(R^n)} \int_{|x_1|\le R'} \int_{R^{n-1}} |\tilde{g} - \tilde{g}_j|(x_1, x')dx'dx_1 \\ &\quad + \|\eta\|_{L^\infty(R^n)} \int_{R^n} |f - f_j| dx \\ &\to 0\end{aligned}$$

as $j \to \infty$ for all $\eta \in C_0^\infty(R^n)$ such that supp $\eta \subset B_{R'}(0)$ for some $R' > 0$. Hence g_j converges weakly to g in $\mathscr{D}'(R^n)$ as $j \to \infty$. We may assume without loss of generality that $g_j(x) \to g(x)$ and $\tilde{g}_j(x) \to \tilde{g}(x)$ a.e. $x \in R^n$. Let

$$E = \{x \in R^n : g_j(x) \to g(x) \text{ and } \tilde{g}_j \to \tilde{g}(x) \text{ as } j \to \infty\},$$

$$E_0 = E \cap \{g < 1\} \cap \left(\bigcap_{j=1}^{\infty} (S(g_j) \cap S(\tilde{g}_j) \cap G(u_j^{(\infty)}, g_j))\right).$$

For any $x_0 \in E_0$, since $g_j(x_0) \to g(x_0)$ as $j \to \infty$, there exists $j_0 \in Z^+$ such that $g_j(x_0) < 1 \ \forall j \ge j_0$. So $g_j(x_0) = f(x_0)$ and $\tilde{g}_j(x_0) = 0$ for all $j \ge j_0$ by Lemma 2.4. Letting $j \to \infty$, we have $g(x_0) = f(x_0)$ and $\tilde{g}(x_0) = 0$. Since $|\{g < 1\} \setminus E_0| = 0$, $g(x_0) = f(x_0)$ and $\tilde{g}(x_0) = 0$ a.e. $x_0 \in \{g < 1\}$ and the theorem follows.

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